Universal enveloping algebra of a set of compatible Lie brackets

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Hamiltonian pairs (or bihamiltonian structures) play an important role in the theory of integrable systems from mathematical physics. Such structures correspond to pairs of compatible Poisson brackets defined on the same manifold. Two Poisson brackets $\{\cdot,\cdot\}_1$ and $\{\cdot,\cdot\}_2$ are said to be compatible if $\alpha\{\cdot,\cdot\}_1 + \beta\{\cdot,\cdot\}_2$ is a Poisson bracket for all $\alpha,\beta \in \Bbbk$, where $\Bbbk$ denotes the ground field. In terms of operads, algebras with compatible Poisson brackets form a so called bi-Hamiltonian operad [1, 2].

In the case of linear Poisson brackets, all such structures arise from a pair of compatible Lie brackets. An algebra $\langle L, [\cdot,\cdot]_1, [\cdot,\cdot]_2, +\rangle$ belongs to a variety Lie_2 of pairs of compatible Lie brackets if $\alpha[\cdot,\cdot]_1 + \beta[\cdot,\cdot]_2$ is a Lie bracket for all $\alpha,\beta \in \Bbbk$.

In [5], the operadic (multiplicative) universal enveloping associative algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$ of a given algebra $\mathfrak{g} \in \mathrm{Lie}_2$ in the sense of V. Ginzburg and M. Kapranov [3] was considered, and the Poincaré—Birkhoff—Witt (PBW) property for it was proved. By the definition, the associative algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$ satisfies the following property: the category of modules over $\mathfrak{g}$ and the category of left modules over $U_{\mathrm{Lie}_2}(\mathfrak{g})$ are equivalent.

In [4], we find the Gröbner—Shirshov basis of the universal enveloping algebra $U_{\mathrm{Lie}_2}(\mathfrak{g}_0)$ of an algebra $\mathfrak{g}_0$, where $\mathfrak{g}_0$ denotes the vector space $\mathfrak{g}$ with both zero Lie brackets. It allows us, applying the PBW property, to get the linear basis of the algebra $U_{\mathrm{Lie}_2}(\mathfrak{g})$. We state that the (exponential) growth rate of this universal enveloping over m-dimensional compatible Lie algebra equals $m+1$.

In a joint work with Zhan Zhen, we extend these results to the case of a Lie algebra of n compatible brackets. Denote $\dim \mathfrak{g} = m$. When $m, n \gg 1$, we prove that the growth rate of $U_{\mathrm{Lie}_n}(\mathfrak{g})$ equals αmn, where $\alpha = \frac{4}{q_0^2} \approx 0,69166$, and q_0 is a minimal root of Bessel function of the first kind.

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Nagata and Keller Automorphisms

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The article is dedicated to study the algebraic automorphisms. Author prove that Nagata polynomial automorphism is algebraic and A Keller algebraic endomorphism is an automorphism.

In paper [1] van der Kulk proved that automorphisms of polynomial ring $K[x, y]$ on two variables are tame.

The group of automorphisms of the algebra $K[x, y]$ admits the structure of an amalgamated free products of the subgroup of affine automorphisms and the subgroup of triangular automorphisms. For the first time, the exact formulation was given by I. R. Shafarevich [2]. D. Wright in his [3] proved an analogue of this result for tame automorphisms $R[x, y]$ over an arbitrary integrity domain R. L. Makar-Limanov showed that automorphisms free associative algebras of rank two over arbitrary fields are tame. L. Makar-Limanov, U. Turusbekova and U.U. Umirbaev proved that automorphisms of free Poisson algebras of rank two over fields of characteristic zero are tame [4] Nagata proved in the article [5] that the automorphism $\sigma = (x + 2y(zx - y^2) + z(zx - y^2), y + z(zx - y^2), z)$ is wild automorphism of the algebra $K[z][x, y]$ over $K[z]$. In our work we study algebraicity of automorphisms.

Theorem 1. *Nagata automorphism is algebraic*

Theorem 2. *A Keller algebraic endomorphism is an automorphism*