Affine registration of point clouds based on point-to-plane approach

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Abstract

The problem of aligning of 3D point data is the known registration task. The most popular registration algorithm is the Iterative Closest Point (ICP). This paper proposes a new algorithm for affine registration of point clouds by incorporating the affine transformation into the point-to-plane ICP algorithm. At each iterative step of the algorithm, a closed-form solution for the affine transformation is derived.

Keywords: affine registration; Iterative closest points (ICP); non-rigid ICP; point-to-plane; surface reconstruction

1. Introduction

The ICP (Iterative Closest Point) algorithm has become the dominant method for aligning of three-dimensional models based purely on geometry. The algorithm is widely used for registering the outputs of 3D scanners. The ICP algorithm starts with two clouds and an initial guess for their relative rigid-body transformation, and then it iteratively refines the transformation by generating pairs of corresponding points on the meshes. The initial alignment may be done by different methods, such as tracking scanner position, identification and indexing of surface features [1, 2], "spin-image" surface signatures [3], computing principal axes of scans [4], exhaustive search for corresponding points [5, 6], or by interactive manner. Since introduction of the ICP [7, 8], many variations of the algorithm have been introduced on the base of the ICP concept.

The ICP algorithm consists of two main stages:

1. Searching of corresponding points (pairs) in two clouds.
2. Minimizing the error metric (variational subproblem of the ICP).

The stage 2 is the key element [12] of ICP. There are two main approaches for the choice of the error metric. The point-to-point method [8] uses the distance between corresponding points in two clouds. The point-to-plane method [7] utilizes the distance between a point in the first cloud and a tangent plane to the second cloud. A solution of the error minimizing problem is based on the affine transformation that brings the best alignment between points of the two clouds. In the orthogonal case the closed-form point-to-point solution was proposed by Horn [9, 10]. The computational complexity of the solution is linear with respect to the number of points.

The traditional ICP algorithm is fast and accurate for the rigid registration between two point clouds but it is unable to handle with the affine case. A modified algorithm of the standard ICP for composition of scaling, rotation, and translation was proposed [11]. A generalized ICP version for an arbitrary affine transformation was suggested [12, 13]. These algorithms are based on the point-to-point approach. A closed-form solution to the point-to-point problem was derived [15-17]. A closed-form solution to the point-to-plane case for orthogonal transformations is unknown. Instead, iterative methods based on the linear least-squares optimization for small angles are often used [14]. In the paper we propose a closed-form solution to the point-to-plane problem for arbitrary affine transformation. The algorithm allows to get the precise solutions of the variational subproblem of the ICP.

2. Formulation of the variational problem

Let $P=\{p^0,\dots,p^{k-1}\}$ be a source point cloud and $Q=\{q^0,\dots,q^{k-1}\}$ be a destination point cloud in $\mathbb{R}^3$. Suppose that the relationship between points in P and Q is given in such a manner that for each point p_i exists a corresponding point q_i. The ICP algorithm is commonly considered as a geometrical transformation for rigid objects mapping P to Q:

$$Rp_i + t, \tag{1}$$

where R is a rotation matrix, t is a translation vector, $i=0,\dots,n-1$. The S-ICP algorithm uses a slightly different geometrical transformation given as

$$RSp_i + t, \tag{2}$$

where S is a scaling matrix.

The group of affine transformations in the dimension of three has 12 generators. It means that the affine transformation in the dimension of three is a function of 12 variables. Let us consider the ICP variational problem for an arbitrary affine transformation in the point-to-plane case. Denote by $S(Q)$ a surface, that is built on the cloud Q, by $T(q^i)$ denote a tangent plane of $S(Q)$ in point q^i. Let $J(A,T)$ be the following function:

$$J(A)=\sum_{i=1}^{n}(< A\,p^i - \; q^i, n^i >)^2, \tag{3}$$

where <·,·> denotes the inner product, A is a matrix of an affine transformation in the homogenous coordinates:

$$A=\begin{pmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

p^i is a point from the cloud P, n^i is the unitary normal for $T(q^i)$:

$$p^i = \begin{pmatrix} p_1^i \\ p_2^i \\ p_3^i \\ 1 \end{pmatrix}, \quad n^i = \begin{pmatrix} n_1^i \\ n_2^i \\ n_3^i \\ 0 \end{pmatrix}. \tag{5}$$

The ICP variational problem can be stated as follows:

$$\arg min_A J(A) \,. \tag{6}$$

3. Closed-form solution to the variational problem

Remark that the function $J(A)$ can be rewritten in the following way:

$$J(A) = \sum_{i=1}^{n} (< A\,p^i - \; q^i, n^i >)^2 = \sum_{i=1}^{n} (< A\,p^i, n^i > - < q^i, n^i >)^2 =$$
$$= \sum_{i=1}^{n} < A\,p^i, n^i >^2 - 2 < A\,p^i, n^i >< q^i, n^i > + < q^i, n^i >^2. \tag{7}$$

The term $< q^i, n^i >$ is a constant with respect to A, thus the variational problem (6) takes the form:

$$\sum_{i=1}^{n} < A\,p^i, n^i >^2 - 2 < A\,p^i, n^i >< q^i, n^i > \to min_a. \tag{8}$$

Let us rewrite the inner product $< A\,p^i, n^i >$ as follows:

$$< A\,p^i, n^i >= \left(a_{11}p_1^i n_1^i + a_{12}p_2^i n_1^i + a_{13}p_3^i n_1^i + t_1 n_1^i\right) + \left(a_{21}p_1^i n_2^i + a_{22}p_2^i n_2^i + a_{23}p_3^i n_2^i + t_2 n_2^i\right) +$$
$$+ \left(a_{31}p_1^i n_3^i + a_{32}p_2^i n_3^i + a_{33}p_3^i n_3^i + t_3 n_3^i\right) + (0). \tag{9}$$

$J(A)$ is a function of 12 variables. Let us consider components of the gradient $\nabla J(A)$.

3.1. Partial derivatives with respect to t

$$\frac{\partial J}{\partial t_j} = \sum_{i=1}^{n} 2 < A\,p^i, n^i > n_j^i - 2n_j^i \; < q^i, n^i > = 0, \;\; j = 1, \ldots, 3. \tag{10}$$

$$\sum_{i=1}^{n} < A\,p^i, n^i > n_j^i - n_j^i \; < q^i, n^i > = \sum_{i=1}^{n} < A\,p^i, n^i > n_j^i - \sum_{i=1}^{n} n_j^i \; < q^i, n^i > = 0. \tag{11}$$

The sum $\sum_{i=1}^{n} n_j^i \; < q^i, n^i >$ is constant with respect to A, denote it by c_j:

$$\sum_{i=1}^{n} n_j^i \; < q^i, n^i >= c_j. \tag{12}$$

The expression (11) takes the form:

$$\sum_{i=1}^{n} < A\,p^i, n^i > n_j^i = c_j, \;\; j = 1, \ldots, 3. \tag{13}$$

Denote by the $(PN)^i$ the following matrix:

$$(PN)^i = p^i\, n^{i\,t} = \begin{pmatrix} p_1^i n_1^i & p_1^i n_2^i & p_1^i n_3^i & 0 \\ p_2^i n_1^i & p_2^i n_2^i & p_2^i n_3^i & 0 \\ p_3^i n_1^i & p_3^i n_2^i & p_3^i n_3^i & 0 \\ n_1^i & n_2^i & n_3^i & 0 \end{pmatrix}. \tag{14}$$

Remark

$$< A\,p^i, n^i >= tr(A \cdot (PN)^i). \tag{15}$$

It follows from

$$< A\,p^i, n^i >=< \begin{pmatrix} a_{11}p_1^i + a_{12}p_2^i + a_{13}p_3^i + t_1 \\ a_{21}p_1^i + a_{22}p_2^i + a_{23}p_3^i + t_2 \\ a_{31}p_1^i + a_{32}p_2^i + a_{33}p_3^i + t_3 \\ 1 \end{pmatrix}, \begin{pmatrix} n_1^i \\ n_2^i \\ n_3^i \\ 0 \end{pmatrix} >=$$

$$= \left(a_{11}p_1^i n_1^i + a_{12}p_2^i n_1^i + a_{13}p_3^i n_1^i + t_1 n_1^i\right) + \left(a_{21}p_1^i n_2^i + a_{22}p_2^i n_2^i + a_{23}p_3^i n_2^i + t_2 n_2^i\right) +$$

$$+\left(a_{31}p_1^in_3^i + a_{32}p_2^in_3^i + a_{33}p_3^in_3^i + t_3n_3^i\right) + (0), \tag{16}$$

and

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1^in_1^i & p_1^in_2^i & p_1^in_3^i & 0 \\ p_2^in_1^i & p_2^in_2^i & p_2^in_3^i & 0 \\ p_3^in_1^i & p_3^in_2^i & p_3^in_3^i & 0 \\ n_1^i & n_2^i & n_3^i & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}p_1^in_1^i + a_{12}p_2^in_1^i + a_{13}p_3^in_1^i + t_1n_1^i & & & \\ & a_{21}p_1^in_2^i + a_{22}p_2^in_2^i + a_{23}p_3^in_2^i + t_2n_2^i & & \\ & & a_{31}p_1^in_3^i + a_{32}p_2^in_3^i + a_{33}p_3^in_3^i + t_3n_3^i & \\ & & & 0 \end{pmatrix}. \tag{17}$$

■

Let us rewrite (13) taking into account (15) as

$$\textstyle\sum_{i=1}^{n} n_j^i \, tr\left(A \cdot (PN)^i\right) = c_j, \; j = 1, \dots, 3. \tag{18}$$

Denote by $(n_jPN)^i$ the following matrices:

$$(n_jPN)^i = \begin{pmatrix} p_1^in_1^in_j^i & p_1^in_2^in_j^i & p_1^in_3^in_j^i & 0 \\ p_2^in_1^in_j^i & p_2^in_2^in_j^i & p_2^in_3^in_j^i & 0 \\ p_3^in_1^in_j^i & p_3^in_2^in_j^i & p_3^in_3^in_j^i & 0 \\ n_1^in_j^i & n_2^in_j^i & n_3^in_j^i & 0 \end{pmatrix}. \tag{19}$$

The following condition holds:

$$n_j^i \, tr\left(A \cdot (PN)^i\right) = tr\left(A \cdot (n_jPN)^i\right). \tag{20}$$

Taking into account (20), we rewrite (18) as

$$\textstyle\sum_{i=1}^{n} tr\left(A \cdot (n_jPN)^i\right) = c_j. \tag{21}$$

Using the property of the matrix trace, we arrive to

$$\textstyle\sum_{i=1}^{n} tr\left(A \cdot (n_jPN)^i\right) = tr\left(\sum_{i=1}^{n} A \cdot (n_jPN)^i\right) = tr\left(A\left(\sum_{i=1}^{n}(n_jPN)^i\right)\right) = c_j. \tag{22}$$

Denote by $M^j, j = 1, \dots, 3$, the following matrices:

$$\textstyle\sum_{i=1}^{n}(n_jPN)^i = M^j, \tag{23}$$

$$M^j = \begin{pmatrix} m_{11}^j & m_{12}^j & m_{13}^j & 0 \\ m_{21}^j & m_{22}^j & m_{23}^j & 0 \\ m_{31}^j & m_{32}^j & m_{33}^j & 0 \\ m_{41}^j & m_{42}^j & m_{43}^j & 0 \end{pmatrix}. \tag{24}$$

The equation (22) can be rewritten as

$$tr\left(A \cdot M^j\right) = c_j, j = 1, \dots, 3. \tag{25}$$

$$A \cdot M^j = \begin{pmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11}^j & m_{12}^j & m_{13}^j & 0 \\ m_{21}^j & m_{22}^j & m_{23}^j & 0 \\ m_{31}^j & m_{32}^j & m_{33}^j & 0 \\ m_{41}^j & m_{42}^j & m_{43}^j & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}m_{11}^j + a_{12}m_{21}^j + a_{13}m_{31}^j + t_1m_{41}^j & & & \\ & a_{21}m_{12}^j + a_{22}m_{22}^j + a_{23}m_{32}^j + t_2m_{42}^j & & \\ & & a_{31}m_{13}^j + a_{32}m_{23}^j + a_{33}m_{33}^j + t_3m_{43}^j & \\ & & & 0 \end{pmatrix}. \tag{26}$$

We get the following equations for $j = 1, \dots, 3$:

$$\left(a_{11}m_{11}^j + a_{12}m_{21}^j + a_{13}m_{31}^j + t_1m_{41}^j\right) + \left(a_{21}m_{12}^j + a_{22}m_{22}^j + a_{23}m_{32}^j + t_2m_{42}^j\right) +$$

$$+\left(a_{31}m_{13}^j + a_{32}m_{23}^j + a_{33}m_{33}^j + t_3m_{43}^j\right) = c_j, j = 1, \dots, 3, i = 1, \dots, n. \tag{27}$$

3.2. Partial derivatives with respect to a

Let us consider the partial derivatives with respect to a_{ij}, $i,j = 1,\dots,3$.

$$\frac{\partial J}{\partial a_{ij}} = \sum_{k=1}^{n} 2 < A\, p^k, n^k > p_j^k n_i^k - 2 p_j^k n_i^k \ < q^k, n^k > = 0, \quad i,j = 1,\dots,3,\ k = 1,\dots,n. \tag{28}$$

$$\sum_{i=1}^{n} < A\, p^k, n^k > p_j^k n_i^k - p_j^k n_i^k \ < q^k, n^k > = \sum_{i=1}^{n} < A\, p^k, n^k > p_j^k n_i^k - \sum_{i=1}^{n} p_j^k n_i^k \ < q^k, n^k > = 0 \tag{29}$$

The sum $\sum_{k=1}^{n} p_j^k n_i^k \ < q^k, n^k >$ is a constant with respect to A, denote it by c_{ij}:

$$\sum_{k=1}^{n} p_j^k n_i^k \ < q^k, n^k > = c_{ij},\ i,j = 1,\dots,3. \tag{30}$$

The expression (29) takes the form:

$$\sum_{k=1}^{n} < A\, p^k, n^k > p_j^k n_i^k = c_{ij}. \tag{31}$$

Taking into account (15), (31) can be rewritten as

$$\sum_{k=1}^{n} p_j^k n_i^k\, tr(A \cdot (PN)^k) = c_{ij},\ i,j = 1,\dots,3. \tag{32}$$

Denote by $(p_j^k n_i^k PN^k)$ the following matrices:

$$(p_j^k n_i^k PN^k) = \begin{pmatrix} p_1^k n_1^k p_j^k n_i^k & p_1^k n_2^k p_j^k n_i^k & p_1^k n_3^k p_j^k n_i^k & 0 \\ p_2^k n_1^k p_j^k n_i^k & p_2^k n_2^k p_j^k n_i^k & p_2^k n_3^k p_j^k n_i^k & 0 \\ p_3^k n_1^k p_j^k n_i^k & p_3^k n_2^k p_j^k n_i^k & p_3^k n_3^k p_j^k n_i^k & 0 \\ n_1^k p_j^k n_i^k & n_2^k p_j^k n_i^k & n_3^k p_j^k n_i^k & 0 \end{pmatrix}. \tag{33}$$

The expression (32) can be rewritten as

$$\sum_{k=1}^{n} tr\left(A \cdot p_j^k n_i^k PN^k\right) = c_{ij}. \tag{34}$$

Denote by M^{ij}, $i,j = 1,\dots,3$, the following matrices:

$$\sum_{k=1}^{n} (p_j^k n_i^k PN^k) = M^{ij}, \tag{35}$$

$$M^{ij} = \begin{pmatrix} m_{11}^{ij} & m_{12}^{ij} & m_{13}^{ij} & 0 \\ m_{21}^{ij} & m_{22}^{ij} & m_{23}^{ij} & 0 \\ m_{31}^{ij} & m_{32}^{ij} & m_{33}^{ij} & 0 \\ m_{41}^{ij} & m_{42}^{ij} & m_{43}^{ij} & 0 \end{pmatrix}. \tag{36}$$

$$AM^j = \begin{pmatrix} a_{11} & a_{12} & a_{13} & t_1 \\ a_{21} & a_{22} & a_{23} & t_2 \\ a_{31} & a_{32} & a_{33} & t_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} m_{11}^{ij} & m_{12}^{ij} & m_{13}^{ij} & 0 \\ m_{21}^{ij} & m_{22}^{ij} & m_{23}^{ij} & 0 \\ m_{31}^{ij} & m_{32}^{ij} & m_{33}^{ij} & 0 \\ m_{41}^{ij} & m_{42}^{ij} & m_{43}^{ij} & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} a_{11}m_{11}^{ij} + a_{12}m_{21}^{ij} + a_{13}m_{31}^{ij} + t_1 m_{41}^{ij} & & & \\ & a_{21}m_{12}^{ij} + a_{22}m_{22}^{ij} + a_{23}m_{32}^{ij} + t_2 m_{42}^{ij} & & \\ & & a_{31}m_{13}^{ij} + a_{32}m_{23}^{ij} + a_{33}m_{33}^{ij} + t_3 m_{43}^{ij} & \\ & & & 0 \end{pmatrix}. \tag{37}$$

Finally, we get 9 equations for $i,j = 1,\dots,3$:

$$\left(a_{11}m_{11}^{ij} + a_{12}m_{21}^{ij} + a_{13}m_{31}^{ij} + t_1 m_{41}^{ij}\right) + \left(a_{21}m_{12}^{ij} + a_{22}m_{22}^{ij} + a_{23}m_{32}^{ij} + t_2 m_{42}^{ij}\right) +$$

$$+\left(a_{31}m_{13}^{ij} + a_{32}m_{23}^{ij} + a_{33}m_{33}^{ij} + t_3 m_{43}^{ij}\right) = c_{ij},\ i,j = 1,\dots,3.. \tag{38}$$

We get the set of 12 linear equations (27), (38) with 12 variables a_{ij}, t_k, $i,j,k = 1,\dots,3$.

Remark

Since $J(A)$ is a convex and bounded function, a solution of the set of equations exists and can be computed by standard SVD (singular value decomposition) method [18].

4. Conclusion

In this paper we considered matching and error minimizing steps of the ICP algorithm. A new efficient algorithm for affine registration of point clouds by incorporating the affine transformation into the point-to-plane ICP algorithm was proposed. At each iterative step of the algorithm, a closed-form solution for the affine transformation was derived.

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References

[1] Faugeras, O. The Representation, Recognition, and Locating of 3-D Objects/ O. Faugeras, M. Hebert // Int. J. Robotic Res. – 1986 -- Vol. 5(3).

[2] Stein, F. Structural Indexing: Efficient 3-D Object Recognition/ F. Stein, G. Medioni// Trans. PAMI – 1992 -- Vol. 14(2).

[3] Johnson, A. Surface Registration by Matching Oriented Points// A. Johnson, M. Hebert/ Proc. 3DIM -- 1997.

[4] Dorai, C. Optimal Registration of Object Views Using Range Data/ C. Dorai, J. Weng, A. Jain// Trans. PAMI -- 1997 -- Vol. 19(10).

[5] Chen, C. A Fast Automatic Method for Registration of Partially-Overlapping Range Images/ C. Chen, Y. Hung, J. Cheng// Proc. ICCV (1998).

[6] Chen, C. RANSAC-Based DARCES: A New Approach to Fast Automatic Registration of Partially Overlapping Range Images/ C. Chen, Y. Hung, J. Cheng // Trans. PAMI – 1999-- Vol. 21(11).

[7] Chen, Y. Object Modeling by Registration of Multiple Range Images/ Y. Chen, and G. Medioni// Proc. IEEE Conf. on Robotics and Automation -- 1991.

[8] Besl, P. A Method for Registration of 3-D Shapes/ P. Besl, N. McKay// Trans. PAMI – 1992 --Vol. 14(2).

[9] Horn, B. Closed-form Solution of Absolute Orientation Using Orthonormal Matrices/ B. Horn, H. Hilden, S. Negahdaripour// Journal of the Optical Society of America -- 1988 -- Series A, 5, 7 -- P.1127-1135.

[10] Horn, B. Closed-Form Solution of Absolute Orientation Using Unit Quaternions/ B. Horn// Journal of the Optical Society of America -- A, 4, 4 – 1987 – P.629–642.

[11] Du, S. An extension of the ICP algorithm considering scale factor/ S. Du, N. Zheng, S. Ying, Q. You, Y. Wu// Proc. 14th IEEE Internat. Conf. on Image Processing (ICIP) – 2007 – P. 193–196.

[12] Du, S. Affine Registration of Point Sets Using ICP and ICA/ S. Du S, N. Zheng, G. Meng, Z. Yuan// IEEE Signal Processing Letters – 2008 – 15 – P. 689-692.

[13] Du, S. Affine iterative closest point algorithm for point set registration/ S. Du, N. Zheng , S. Ying, J. Liu// Pattern Recognition Letters –2010 – 31 -- P.791–799.

[14] Rusinkiewicz, S. Efficient Variants of the ICP Algorithm/ S. Rusinkiewicz, M. Levoy// Proceedings of the International Conference on 3-D Digital Imaging and Modeling – 2001 – P.145–152.

[15] Tihonkih, D. A modified iterative closest point algorithm for shape registration/ D. Tihonkih, A. Makovetskii, V. Kuznetsov// Proc. SPIE Applications of Digital Image Processing XXXIX -- 2016 -- Vol. 9971 -- 99712D.

[16] Tihonkih, D. The iterative closest points algorithm and affine transformations/ D. Tihonkih, A. Makovetskii , V. Kuznetsov// Proc. Int. Conference of Analysis of Images, Social Networks, and Texts (AIST 2016) – 2016 -- P. 351-359.

[17] Vokhmintsev, A. A fusion algorithm for building three-dimensional maps/ A. Vokhmintsev, A. Makovetskii, V. Kober, I. Sochenkov, V. Kuznetsov// Proc. SPIE's 60 Annual Meeting: Applications of Digital Image Processing XXXVIII – 2015 -- Vol. 9599 -- P. 959929-1.

[18] Press, W. H. Numerical Recipes in C: The Art of Scientific Computing, Second Edition/ W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery -- Cambridge University Press, 1992.