Classification of symmetric Leibniz algebras associated to quasi-filiform Lie algebras

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In recent years, the theory of Leibniz algebras has been intensively studied and many results on Lie algebras have been extended to Leibniz algebras. A left (right) Leibniz algebra is a non-associative algebra where the left (right) multiplications are derivations. Symmetric Leibniz algebra is an algebra which is simultaneously left and right Leibniz algebra. The initial property and theory of symmetric Leibniz algebras are given in the works of Benayadi and Hidri [2]. They gave a method for the classification of symmetric Leibniz algebras, which is based on the property that a symmetric Leibniz algebra forms a Poisson algebra with respect to the commutator and anticommutator [1].

Using this method, the classification of symmetric Leibniz algebras underlying Lie algebra is a naturally-graded filiform Lie algebras $\mathbf{n}_{n,1}$ and $\mathcal{Q}_{2n}$ is obtained in [3]. Moreover, the classification of 5-dimensional symmetric Leibniz algebras is given in [4]. In this work, we focus on the classification of symmetric Leibniz algebras associated with Lie naturally-graded quasi-filiform Lie algebras.

Definition 1 An algebra $(\mathcal{L}, [-,-])$ over a field F is called Lie algebra if for any $x, y, z \in \mathcal{L}$ the following identities hold:

$$[x, y] = -[y, x], \quad [x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

Definition 2 An algebra $(\mathcal{L}, \cdot)$ is said to be a symmetric Leibniz algebra, if for any $x, y, z \in \mathcal{L}$ it satisfies the following identities:

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + y \cdot (x \cdot z), \quad (x \cdot y) \cdot z = x \cdot (y \cdot z) + (x \cdot z) \cdot y.$$

Let $\mathcal{L}$ be a vector space equipped with a bilinear map $\cdot : \mathcal{L} \times \mathcal{L} \to \mathcal{L}$. For all $x, y \in \mathcal{L}$, we define [-,-] and $\circ$ as follows

$$[x, y] = \frac{1}{2}(x \cdot y - y \cdot x), \qquad x \circ y = \frac{1}{2}(x \cdot y + y \cdot x).$$

Proposition 1 [1]. Let $(\mathcal{L}, \cdot)$ be an algebra. The following assertions are equivalent:

1. $(\mathcal{L}, \cdot)$ is a symmetric Leibniz algebra.

2. The following conditions hold:

   (a) $(\mathcal{L}, [-,-])$ is a Lie algebra.

   (b) For any $u, v \in \mathcal{L}$, $u \circ v$ belongs to the center of $(\mathcal{L}, [-,-])$.

   (c) For any $u, v \in \mathcal{L}$, $([u,v]) \circ w = 0$ and $(u \circ v) \circ w = 0$.

According to this Proposition, we derive that any symmetric Leibniz algebra is given by a Lie algebra $(\mathcal{L}, [-,-])$ and a symmetric bilinear form $\omega : \mathcal{L} \times \mathcal{L} \to Z(\mathcal{L})$, where $Z(\mathcal{L})$ is the center of the Lie algebra, such that for any $x, y, z \in \mathcal{L}$

$$\omega([x,y], z) = \omega(\omega(x,y), z) = 0.$$

Then the product $u \cdot v = [u,v] + \omega(u,v)$ gives a symmetric Leibniz algebra structure.

In the following Proposition the criteria of isomorphism of two symmetric Leibniz algebras with the symmetric bilinear forms ω and μ is given.

Proposition 2[1] Let $(\mathcal{G}, [-,-])$ be a Lie algebra and ω and μ two solutions of (). Then $(\mathcal{G}, \cdot_\omega)$ is isomorphic to $(\mathcal{G}, \cdot_\mu)$ if and only if there exists an automorphism A of $(\mathcal{G}, [-,-])$ such that

$$\mu(u,v) = A^{(-1)}\omega(Au, Av).$$

For an arbitrary symmetric Leibniz algebra $(\mathcal{L}, \cdot)$, we define the series:

$$\mathcal{L}^1 = \mathcal{L}, \ \mathcal{L}^{k+1} = \mathcal{L}^k \cdot \mathcal{L}, \quad k \geq 1.$$

Definition 3 A n-dimensional symmetric Leibniz algebra $\mathcal{L}$ is called *nilpotent* if there exists $k \in N$ such that $\mathcal{L}^k = 0$. Such minimal number is called *index of nilpotency*.

A n-dimensional symmetric Leibniz algebra with index of nilpotency n and $n-1$ is called filiform and quasi-filiform, respectively.

Definition 4 Given a nilpotent Lie algebra $\mathcal{L}$, with index of nilpotency s. Put $\mathcal{L}_i = L^i/\mathcal{L}^{i+1}$, $1 \leq i \leq s-1$, and denote $Gr(\mathcal{L}) = \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \cdots \oplus \mathcal{L}_{n-1}$. Define the product in $Gr(\mathcal{L})$ as follows:

$$[x + \mathcal{L}^{i+1}, y + \mathcal{L}^{j+1}] := [x,y] + \mathcal{L}^{i+j+1},$$

where $x \in \mathcal{L}^i/\mathcal{L}^{i+1}, y \in \mathcal{L}^j/\mathcal{L}^{j+1}$. Then $[\mathcal{L}_i, \mathcal{L}_j] \subseteq \mathcal{L}_{i+j}$ and we obtain the graded algebra $Gr(\mathcal{L})$.If $Gr(\mathcal{L})$ and $\mathcal{L}$ are isomorphic, then we say that the algebra $\mathcal{L}$ is naturally-graded.

The complete algebraic classification of naturally-graded quasi-filiform Lie algebras was given in [5]. In this work, we give the description of all symmetric Leibniz algebras whose underlying Lie algebra is a naturally-graded quasi-filiform Lie algebra.

References

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About geometry of completely integrable Hamiltonian systems

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We are interested in geometry of Liouville foliation generated by completely integrable Hamiltonian systems.

Definition 1 [2] Let M^{2n} be a symplectic manifold and $sgradH$ Hamiltonian vector field with a smooth Hamiltonian function H.

Hamiltonian system $sgradH$ is called *completely integrable in the sense of Liouville or completely integrable*, if exists set of smooth functions $f_1, ..., f_n$ as:

1) $f_1, ..., f_n$ are first integrals of $sgradH$ Hamiltonian vector field,

2) they are functionally independent on M, that is, almost everywhere on M their gradients are linearly independent,

3) $\{f_i, f_j\} = 0$ for any i and j,

4) the vector fields $sgradf_i$ are complete, that is natural parameter on their integral trajectories is defined on the whole number line.

Definition 2 [1] Partition of the manifold M^m into connected components of joint level surfaces of the integrals $f_1, ..., f_n$ is called *The Liouville foliation* corresponding to the completely integrated system.