The WKB 4x4 method for an inhomogeneous chiral layer

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Abstract. The propagation of a plane electromagnetic wave in a plane inhomogeneous chiral medium with oblique incidence is considered. A solution of the ordinary differential equations system 4x4 was obtained by the Wentzel–Kramers–Brillouin method in the form of a Cauchy matrix. The dependence of the cross-polarized components on the reflection from the profile of the chirality parameter and the angle of incidence is shown.

Keywords: planar structures, chiral medium, reflection matrix, WKB method, 4x4 method, eigenwaves.

1. Introduction

Interest in chiral media is called problem solving nanophotonics: superresolution effect, overcoming the diffraction limit [1]. The first work on chiral media studied the optical activity of quartz [2]. The first artificial material possessing the chirality property was described in the work of Lindman [3] in 1920. Interest in chiral media has increased significantly after the release of [4]. It was investigated by the method of integral equations of scattering of electromagnetic waves on asymmetric spheroid objects. Fresnel [5] as the basis used two waves with different phase velocities right and left circular polarization (RCP and LCP). Such a representation is a traditional [6] and is found in the works of various researchers. The problem of propagation of electromagnetic waves in the chiral medium is reduced to a system of four ordinary differential equations (ODE), which explains the interaction between waves of different polarizations. Dissemination of TE wave in a chiral medium leads to the appearance of TH waves and vice versa. Upon reflection of each of the waves on the border there are two waves: TH and TE, or s- and p- polarization. The waves RCP and LCP in the medium and the reflection from the interfaces interact. Basic solutions for a chiral medium should be chosen so that they are linearly independent. Therefore, as a basic wave chiral medium is advisable to choose the eigenvectors of the system of ODE corresponding to the eigenvalues. Were found waves with elliptical polarization, and not interacting. They are conveniently used as the base. Each resulting basic solution corresponds to one of four eigenvalues ODE.

2. Equations

Consider the oblique incidence of a plane electromagnetic wave (EMW) on a planar inhomogeneous chiral layer, shown in Figure 1a. The light falls at an angle θ_i, is reflected at an angle θ_r, refracted and propagates further in the layer at an angle $\theta_t(z)$, that depends on the z coordinate. In deriving the matrix of the solution for the layer, we use the material equations of the chiral medium [7]:

$$\vec{D} = \varepsilon\vec{E} \mp i\chi\vec{H},$$
$$\vec{B} = \mu\vec{H} \pm i\chi\vec{E}. \quad (1)$$

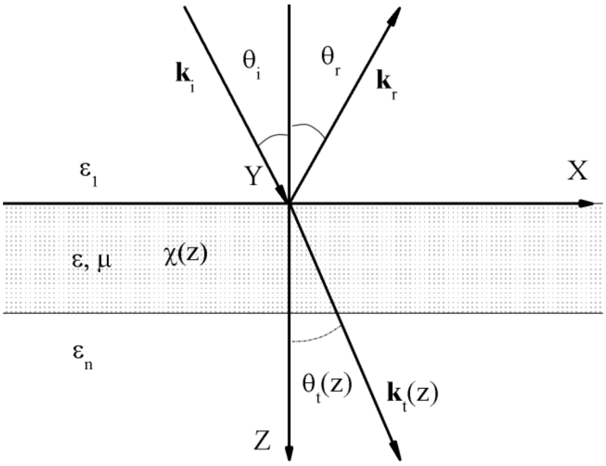
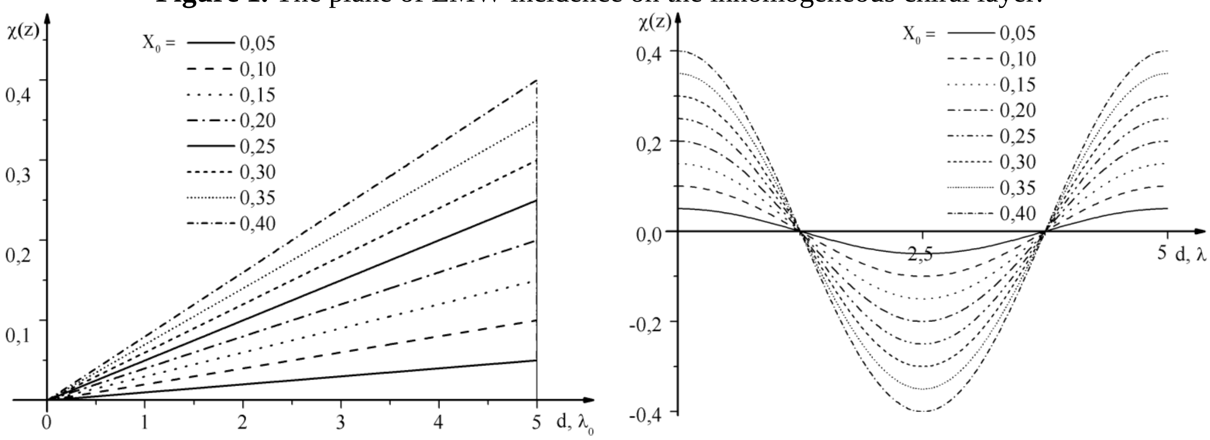


Figure 1. The plane of EMW incidence on the inhomogeneous chiral layer.



a) b)

Figure 2. The profile of the function χ(z): a) by formula (2.a), b) by formula (2.b).

From Snell's law it follows that $\sqrt{\varepsilon_1}\sin\theta_i = \left(\sqrt{\varepsilon(z)} \pm \chi\right)\sin\theta_t(z) = \sqrt{\varepsilon_n}\sin\theta_n$. We introduce the notation: $\alpha = \sqrt{\varepsilon_1}\sin\theta_i$. We consider two cases: the first, when the chirality of the medium depends linearly on the coordinate z

$$\chi(z) = z \cdot \frac{X_0}{d}, \quad (2.a)$$

and the second, when the chirality has the form of a function:

$$\chi(z) = \frac{X_0}{d}\cos\left(\frac{2\pi}{d}z\right). \quad (2.b)$$

The values X_0 correspond to the graphs shown in Figure 2. A system of four ODEs for an inhomogeneous chiral layer with allowance for (1) in Cartesian coordinates has the form:

$$\frac{d}{dz}\vec{Q} = ik_0\hat{A}Q. \quad (3.a)$$

Here $k_0 = \dfrac{2\pi}{\lambda_0}$ - is the wave number, and λ_0 - is the wavelength. So an expression for the matrix $\hat{A}$ is:

$$\hat{A} = \begin{pmatrix} 0 & -\mu & 0 & \mp i\chi \\ -\left(\varepsilon - \dfrac{\varepsilon\alpha^2}{\varepsilon\mu - \chi^2}\right) & 0 & \pm i\chi\left(1 + \dfrac{\alpha^2}{\varepsilon\mu - \chi^2}\right) & 0 \\ 0 & \mp i\chi & 0 & \varepsilon \\ \pm i\chi\left(1 + \dfrac{\alpha^2}{\varepsilon\mu - \chi^2}\right) & 0 & \mu - \dfrac{\mu\alpha^2}{\varepsilon\mu - \chi^2} & 0 \end{pmatrix}. \tag{3.b}$$

It describes the process of changing the projection of waves propagating in the chiral layer. We write it more briefly:

$$\hat{A} = \begin{pmatrix} 0 & -b & 0 & q \\ -c & 0 & p & 0 \\ 0 & q & 0 & e \\ p & 0 & f & 0 \end{pmatrix}. \tag{4.a}$$

Or in the form of four blocks - submatrices:

$$\hat{A} = \begin{pmatrix} \hat{A}_s & \hat{B} \\ \hat{B} & \hat{A}_p \end{pmatrix}. \tag{4.b}$$

The matrix blocks $\hat{A}_s$, $\hat{A}_p$, $\hat{B}$ have dimension $\dim \hat{A}_s = \dim \hat{A}_p = \dim \hat{B} = 2$. Submatrix $\hat{A}_s$ describes the change in the medium of a wave component having s-polarization (TE), $\hat{A}_p$ - is the change in the p-polarization component (TM), and, $\hat{B}$ - describes the interaction of s- and p-polarizations.

The vector $\vec{Q}$ is made up of the projections of the fields $\vec{E}$ and $\vec{H}$: $\vec{Q} = \begin{pmatrix} E_y & H_x & H_y & E_x \end{pmatrix}^T$. To find the matrix solution, we find the eigenvalues and eigenfunctions of the matrix $\hat{A}$. The eigenvalues are solutions of the characteristic equation $\det\left(\hat{A} - \lambda\hat{I}\right) = 0$ and are calculated by formulas:

$$\lambda_{1,2} = \pm\frac{1}{\sqrt{2}}\sqrt{bc + ef + 2pq - \sqrt{\left(bc - ef\right)^2 - 4bp\left(ep - cq\right) + 4fq\left(ep - cq\right)}}, \tag{5.a}$$

$$\lambda_{3,4} = \pm\frac{1}{\sqrt{2}}\sqrt{bc + ef + 2pq + \sqrt{\left(bc - ef\right)^2 - 4bp\left(ep - cq\right) + 4fq\left(ep - cq\right)}}. \tag{5.b}$$

Since we start from the material equations (1), the following equalities hold: $bc = ef$, $ep - cq = \pm 2i\varepsilon\chi$, $fq - bp = \mp 2i\mu\chi$, so the form of the eigenvalues (5.a) –(5.b) is essentially simplified and, according to [8], reduces to the form:

$$\lambda_{1,2} = \sqrt{\left(\sqrt{\varepsilon\mu} - \chi\right)^2 - \alpha^2}, \tag{6.a}$$

$$\lambda_{1,2} = \sqrt{\left(\sqrt{\varepsilon\mu} + \chi\right)^2 - \alpha^2}. \tag{6.b}$$

The basic functions for Wentzel–Kramers–Brillouin approximation (WKB) as $S_i \exp\left(ik_0\int\limits_0^z \lambda_i\left(\xi\right)d\xi\right)$ define the fundamental matrix of the solution (FMS) $\hat{Y}\left(z\right)$, as was shown in W. Wazov's book [9]:

$$\hat{Y}(z)=\hat{F}(z)diag\left(\exp\left(ik_0\int_0^z\lambda_1(\xi)d\xi\right),....,\quad \exp\left(ik_0\int_0^z\lambda_4(\xi)d\xi\right)\right). \quad (7)$$

To find the FMS, we first write down the components of the fields E_y and H_y:

$$E_y=S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi}+S_2e^{ik_0\int_0^z\lambda_1(\xi)d\xi}, \quad (8)$$

$$H_y=S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi}+S_4e^{ik_0\int_0^z\lambda_4(\xi)d\xi}. \quad (9)$$

From the first and third equations of the system (2) with allowance for the matrix (3), we obtain the relations between the projections of the fields of the EMW:

$$\begin{cases} qE_x-bH_x=\lambda_1S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi}-\lambda_2S_2e^{ik_0\int_0^z\lambda_2(\xi)d\xi}, \\ eE_x+qH_x=\lambda_3S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi}-\lambda_4S_4e^{ik_0\int_0^z\lambda_4(\xi)d\xi}. \end{cases} \quad (10)$$

We solve the system (10) using the Cramer rule. The determinant of the matrix of system (10) in the plane z=const:

$$\Delta=eb+q^2=\varepsilon\mu-\chi^2(z), \quad (11)$$

Determinants Δ_1 and Δ_2:

$$\Delta_1=\lambda_1q\left(S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi}-S_2e^{-ik_0\int_0^z\lambda_1(\xi)d\xi}\right)+\lambda_3b\left(S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi}-S_3e^{-ik_0\int_0^z\lambda_3(\xi)d\xi}\right). \quad (12)$$

$$\Delta_2=-\lambda_1e\left(S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi}-S_2e^{-ik_0\int_0^z\lambda_1(\xi)d\xi}\right)+\lambda_3q\left(S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi}-S_3e^{-ik_0\int_0^z\lambda_3(\xi)d\xi}\right). \quad (13)$$

$$E_x=\frac{\Delta_1}{\varepsilon\mu-\chi^2}, \quad (14.a)$$

$$H_x=\frac{\Delta_2}{\varepsilon\mu-\chi^2}. \quad (14.b)$$

We write down all the projections of the fields in the FMS lines:

$$Y(z)=\begin{pmatrix} S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi} & S_2e^{-ik_0\int_0^z\lambda_1(\xi)d\xi} & 0 & 0 \\ \frac{-e\lambda_1}{\varepsilon\mu-\chi^2}S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi} & \frac{e\lambda_1}{\varepsilon\mu-\chi^2}S_2e^{-ik_0\int_0^z\lambda_1(\xi)d\xi} & \frac{q\lambda_3}{\varepsilon\mu-\chi^2}S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi} & -\frac{q\lambda_3}{\varepsilon\mu-\chi^2}S_3e^{-ik_0\int_0^z\lambda_3(\xi)d\xi} \\ 0 & 0 & S_3e^{ik_0\int_0^z\lambda_1(\xi)d\xi} & S_4e^{-ik_0\int_0^z\lambda_3(\xi)d\xi} \\ \frac{q\lambda_1}{\varepsilon\mu-\chi^2}S_1e^{ik_0\int_0^z\lambda_1(\xi)d\xi} & -\frac{q\lambda_1}{\varepsilon\mu-\chi^2}S_2e^{-ik_0\int_0^z\lambda_1(\xi)d\xi} & \frac{b\lambda_3}{\varepsilon\mu-\chi^2}S_3e^{ik_0\int_0^z\lambda_3(\xi)d\xi} & -\frac{b\lambda_3}{\varepsilon\mu-\chi^2}S_3e^{-ik_0\int_0^z\lambda_3(\xi)d\xi} \end{pmatrix} \quad (15)$$

The values of λ_i and χ in the matrix $Y(z)$ are determined in an arbitrary plane z=const inside, or on any boundary layer with an applicator z. The Cauchy matrix is calculated from the found FMS:

$$\hat{N}(z,0)=\hat{Y}(z)Y^{-1}(0). \quad (16)$$

The matrix $\hat{Y}^{-1}(0)$ - is the inverse of the FMS matrix written on the boundary z=0. The Cauchy matrix (16) makes it possible to stitch the boundary conditions in a matrix form. From the continuity conditions for the tangential components of the fields $\vec{E}$ and $\vec{H}$ at the interfaces of the media, we obtain a system connecting the s- and p- components of the incident and reflected waves:

$$\begin{pmatrix} p_n\left(n_{11}+p_1n_{12}\right)-\left(n_{21}+p_1n_{22}\right) & \sqrt{\varepsilon_1}\left(p_n\left(n_{13}+q_1n_{14}\right)-\left(n_{23}+q_1n_{24}\right)\right) \\ \sqrt{\varepsilon_n}\left(q_n\left(n_{31}+p_1n_{32}\right)-\left(n_{41}+p_1n_{42}\right)\right) & \sqrt{\varepsilon_1\varepsilon_n}\left(q_n\left(n_{33}+q_1n_{34}\right)-\left(n_{43}+q_1n_{44}\right)\right) \end{pmatrix}\begin{pmatrix} E_{si} \\ E_{pi} \end{pmatrix}=$$
$$=\begin{pmatrix} -\left(p_n\left(n_{11}-p_1n_{12}\right)-\left(n_{21}-p_1n_{22}\right)\right) & -\sqrt{\varepsilon_1}\left(p_n\left(n_{13}-q_1n_{14}\right)-\left(n_{23}-q_1n_{24}\right)\right) \\ -\sqrt{\varepsilon_n}\left(q_n\left(n_{31}-p_1n_{32}\right)+\left(n_{41}-p_1n_{42}\right)\right) & -\sqrt{\varepsilon_1\varepsilon_n}\left(q_n\left(n_{33}-q_1n_{34}\right)-\left(n_{43}-q_1n_{44}\right)\right) \end{pmatrix}\begin{pmatrix} E_{sr} \\ E_{pr} \end{pmatrix}. \tag{17}$$

Where n_{ij} - are the elements of the Cauchy matrix. The matrix equation (17) can be briefly written in the form:

$$\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}\begin{pmatrix} E_{si} \\ E_{pi} \end{pmatrix}=\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}\begin{pmatrix} E_{sr} \\ E_{pr} \end{pmatrix}. \tag{18}$$

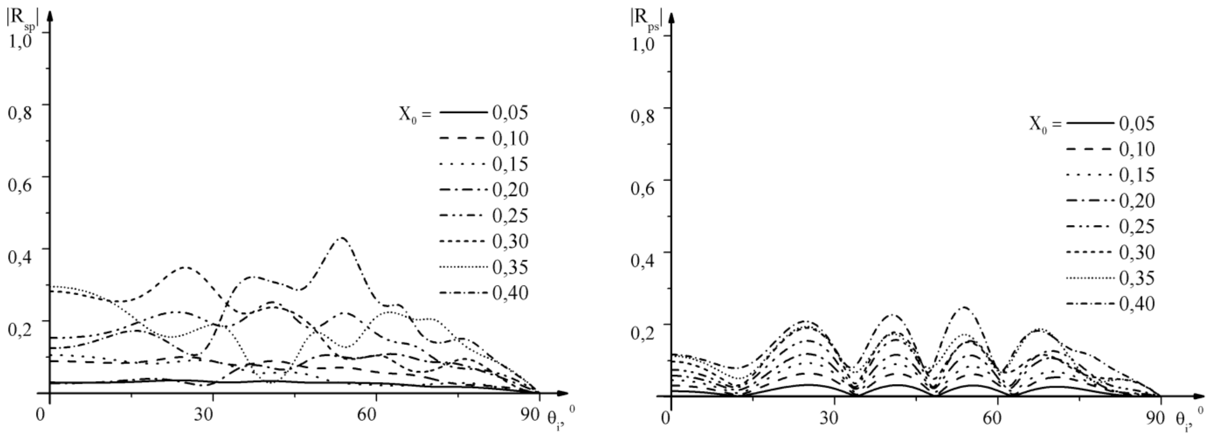
Then the matrix of reflection coefficients for s- and p-polarization waves has the form:

$$\hat{R}=\begin{pmatrix} R_{ss} & R_{sp} \\ R_{ps} & R_{pp} \end{pmatrix}=\frac{1}{\Delta U}\begin{pmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{pmatrix}\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}. \tag{19}$$

Upon reflection from the chiral medium, the polarization of light changes: if a wave of s polarization falls on the layer, then the reflected wave will have an s and p component. Analogous transformations occur when a p wave is incident on the chiral layer.

3. Calculation of the reflection matrix at the boundary of the inhomogeneous chiral layer

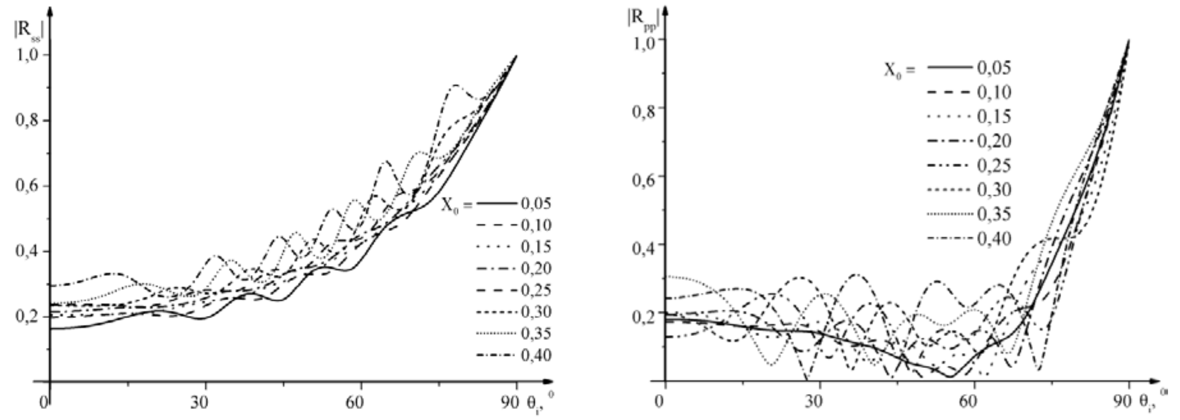
Let us find out whether the value of the chirality parameter and its inhomogeneous medium profile affect the absolute values of the matrix coefficients (19). Consider an artificial medium of thickness $d=5\lambda_0$, in which $\varepsilon=2$, χ varies, according to (2a), a X_0 a takes on the values $\mathrm{X}_0=0{,}05$, $0{,}10$, …, $0{,}40$, $\varepsilon_n=1.7782$. The results of the calculations are presented in Figures 3 и 4. As X_0 increases, the growth of $|R_{sp}|$ and $|R_{ps}|$ a increases for $|R_{ss}|$ and $|R_{pp}|$ the interference in the layer increases**.**



a) b)

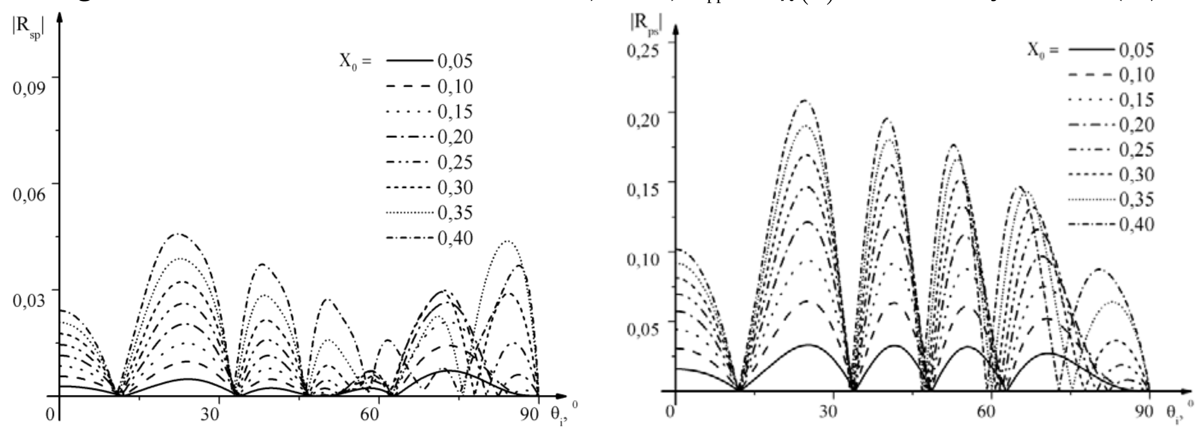
Figure 3. Modules of reflection coefficients a) R_{sp}, b) R_{ps} for $\chi(z)$ calculated by formula (2a).

For comparison, we considered a layer with the same parameters, $d=5\lambda_0$, but the profile of $\chi(z)$, had the form (2b). The modules of the coefficients of the matrix $\hat{R}$ were calculated. The results are shown in Figure 5(a-b) and Figure 6(a-b). Similarly, as X_0 increases $|R_{sp}|$ and $|R_{ps}|$ increases for $|R_{ss}|$ and $|R_{pp}|$ increases the interference in the layer. If the value of X_0 increases, then the values of $|R_{sp}|$ and $|R_{ps}|$ increase. According to the graphs of $|R_{ss}|$ and $|R_{pp}|$, it is seen that under these conditions interference in the layer increases.



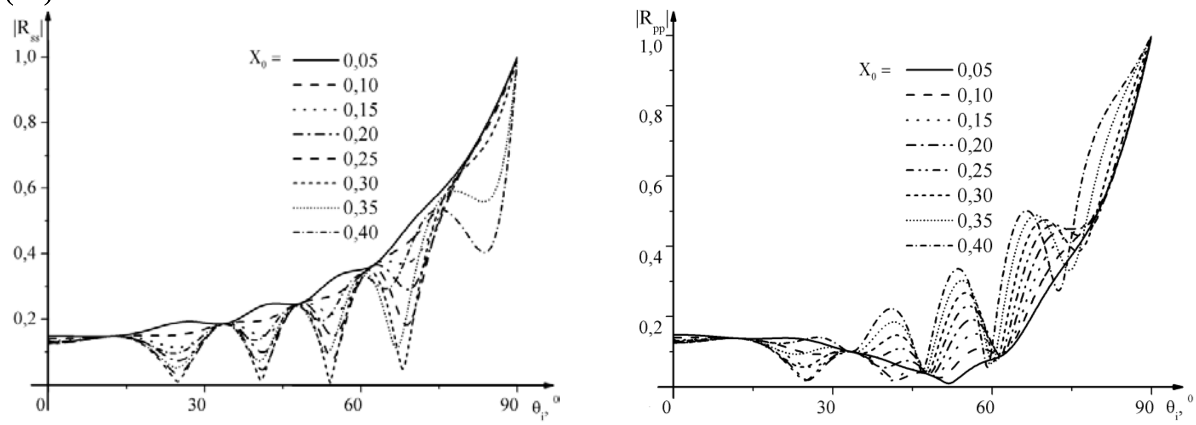
a) b)

Figure 4. Modules of reflection coefficients a) R_{ss}, b) R_{pp} for $\chi(z)$ calculated by formula (2a).



a) b)

Figure 5. Absolute values of the reflection coefficients a) R_{sp}, b) R_{ps} for $\chi(z)$ calculated by formula (2b).



a) b)

Figure 6. Absolute values of the reflection coefficients a) R_{ss}, b) R_{pp} for $\chi(z)$ varying by the formula (2b).

In the present paper, we obtain a matrix solution for an inhomogeneous chiral layer, which makes it possible to calculate the field vectors of the EMW in the medium, as well as the reflection matrices. It is shown that the maximum value and the profile of the chirality parameter determine the cross-polarization of the EMW under reflection. The resulting mathematical apparatus will allow to design new polarization devices, optical switches.

4. References

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