Non-Markovian dynamics of a three-level quantum system

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Abstract

We investigate non-Markovian dynamics of a three-level system with the help of stochastic Schrödinger equation (SSE). The SSE has significant advantages against other approaches to open quantum systems. First of all, the dimension of the SSE is much smaller than the dimension of the corresponding master equation. Second, SSE ensures the complete positivity of the reduced density operator in non-Markovian case, that is hard to achieve using other approaches. The above mentioned facts open a new efficient way for study quantum systems in higher dimensions in both Markovian and non-Markovian regimes.

Keywords: stochastic Schrödinger equation; three-level systems; non-Markovian dynamics

1. Introduction

Open quantum systems are usually described by the reduced density operator, which satisfies a master equation. The master equations can be strictly divided into two classes: Markovian and non-Markovian [1]. The Markovian master equations traditionally represent systems of the first order differential equations with the constant coefficients and they are well studied at present. In opposite, non-Markovian master equations forsake many open questions and they are intensively studied during last several years [2]. It is clear today that the non-Markovian master equations may be of two types either integro-differential or diferential with variable coefficients. Both types of the non-Markovian master equations are equivalent and the differential one is used more often. Unfortunately, the general form of the non-Markovian master equation, which ensures the complete positivity of the density operator is still unknown.

Another approach to describe dynamics of open quantum systems is to use Stochastic Schrödinger equation (SSE) for the wave vector driven by the noise [3, 4]. The reduced density operator is recovered by mean of stochastic averaging over many such vectors. There is an exact correspondence between Markovian master equations and SSEs. It is clear that the dimension of the wave vector is smaller than the dimension of the reduced density operator. This fact open a new possibilities for investigation of high-dimensional open quantum systems, such as spin chains, photosynthetic reaction centre, etc.

As in the case of non-Markovian master equations, non-Markovian SSEs are also intensively examined. The main attempts of researchers here are focused on the so called unravelling of the non-Markovian master equations [5], i.e. construction of a SSE which reproduces all the results given by the master equation. It is not always possible especially for integro-differential master equations. On the other hand, one can generalise Markovian SSEs to non-Markovian ones without direct connection with the master equation formalism. Such an approach has many advantages and one of them is ensuring of complete positivity of the reduced density operator.

In this paper we consider the non-Markovian generalization of the SSE for a three-level quantum system. We present results of the direct simulation of the SSE for this model and discuss the main difference between Markovian and non-Markovian SSEs.

2. Model

Three level systems can be of three types Λ, V and cascade. The difference between them is forbidden transitions between the energy levels. Let us consider the three-level system of V -types for concreteness. The Hamiltonian of the V system in the photonic thermostat is [6] (we set $\hbar = 1$)

$$H = \omega_0 H_1 + \Omega_0 H_2 + \sum_{j=1}^{\infty} \omega_j b_j^\dagger b_j + \sum_{i=1}^{\infty} [(f_i J_+ + g_i L_+) b_i + h.c.], \tag{1}$$

where b_j and $b_j^\dagger$ are creation and annihilation operator of the j th photon in the thermostate with frequency ω_j, $\omega_0/2$ and $\Omega_0 + \omega_0/2$ are transition frequency in the V- system, f_i and g_i are the constant of system-photon interaction and the matrices H_1, H_2, J_+, L_+ are define as follow [6]

$$H_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, H_2 = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, J_+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, K_+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{2}$$

2.1. Markovian evolution

The Markovian master equation for the reduced density operator can be written as [6]

$$\frac{\partial\rho}{\partial t} = \frac{\gamma_J}{2}[(N_J+1)(2J_-\rho J_+ - J_+J_-\rho - \rho J_+J_-) + N_J(2J_+\rho J_- - J_-J_+\rho - \rho J_-J_+)] + (J \leftrightarrow K), \tag{3}$$

where $\gamma_{J,K}$ is the damping constant, $N_{J,K}$ is average number of the heat photons on the corresponding transition.

The standard procedure of unravelling allows to write the SSE, corresponding to (3). The SSE has the following form

$$d|\psi\rangle = -\frac{\gamma_J}{2}\left((N_J+1)J_+J_- + N_JJ_-J_+\right)|\psi\rangle dt + i\sqrt{\gamma_J(N_J+1)}J_-|\psi\rangle dW_J^1 + i\sqrt{\gamma_JN_J}J_+|\psi\rangle dW_J^2 + (J \leftrightarrow K), \tag{4}$$

where $W_{J,K}^{1,2}$ are independent standard Wiener processes. It is easy to verify using Ito calculus that $\rho = \mathbb{E}(|\psi\rangle\langle\psi|)$ satisfies the master equation (3).

2.2. Non-Markovian evolution

To describe non-Markovian effects Barchielli in [7] suggested to replace Markovian Wiener processes in (4) by some non-Markovian noises. The simplest non-Markovian noise is the Ornstein-Uhlenbeck one, which is satisfyes the stochastic equation

$$dX = -kXdt + dW, \tag{5}$$

where $k > 0$ is the constant. By substitution the Ornstein-Uhlenbeck processes (5) instead of the Wiener increments in (4) we derived the following non-Markovian SSE

$$\begin{aligned} d|\tilde\psi\rangle = -\left(\frac{\gamma_J}{2}(N_J+1)J_+J_- + \frac{\gamma_J}{2}N_JJ_-J_+ + ik_J^1X_J^1\sqrt{\gamma_J(N_J+1)}J_- + ik_J^2X_J^2\sqrt{\gamma_JN_J}J_+\right)|\tilde\psi\rangle dt \\ +i\sqrt{\gamma_J(N_J+1)}J_-|\tilde\psi\rangle dW_J^1 + i\sqrt{\gamma_JN_J}J_+|\tilde\psi\rangle dW_J^2 + (J \leftrightarrow K). \end{aligned} \tag{6}$$

Unfortunately, the above SSE is not a mean-1 martingale and we have to somehow modify the equation to satisfy the martingale property. It is straightforward to check that to be a mean-1 martingale Eq. (7) needs 4 more terms in the drift part, namely

$$\begin{aligned} d|\tilde\psi\rangle = -\left(\frac{\gamma_J}{2}(N_J+1)J_+J_- + \frac{\gamma_J}{2}N_JJ_-J_+ + ik_J^1X_J^1\sqrt{\gamma_J(N_J+1)}(J_- + J_+) + ik_J^2X_J^2\sqrt{\gamma_JN_J}(J_- + J_+)\right)|\tilde\psi\rangle dt \\ +i\sqrt{\gamma_J(N_J+1)}J_-|\tilde\psi\rangle dW_J^1 + i\sqrt{\gamma_JN_J}J_+|\tilde\psi\rangle dW_J^2 + (J \leftrightarrow K). \end{aligned} \tag{7}$$

Obviously, that Eq. (7) is transformed to Eq. (4) when all $k_m^l = 0$. Moreover, it is easy to prove that Eq. (7) is a martingal, i.e. $\mathbb{E}(\langle\tilde\psi(t)|\tilde\psi(t)\rangle) = 1$ and $\tilde\rho(t) = \mathbb{E}(|\tilde\psi(t)\rangle\langle\tilde\psi(t)|)$ is a completely positive operator by construction. Note, that operator $\tilde\rho(t)$ does not satisfy the Markovian master equation (3) and, even more, we cannot construct any closed master equation for this operator due to the presence of the noise terms in the drift part of the equation (7). All the above mentioned facts demonstrate the uniqueness of Eq. (7).

3. Results of simulation

The Non-Markovian SSE (7) can be efficiently simulated. It is possible to do if we add to the three components of the wave vector $|\tilde\psi\rangle$, four stochastic equations for the Ornstein-Uhlenbeck noises. Resulting seven equations form a closed system and may be numerically solved by any suitable algorithm. The initial values for the Ornstein-Uhlenbeck processes are normally distributed random numbers with zero mean and unit standard deviation.

The results of simulation are presented in Fig. 1. The simulation was performed with the help of the stochastic Euler algorithm [8] and the results was averaged over 10^4 realizations. The error bars are also included in the graphic. In the same pictures we draw the dynamics given by the Markovian master equation (3). One can see that the non-Markovian noise has significant effect on dynamics and cannot be neglected.

4. Conclusion

In this paper we have derived the non-Markovian SSE for a three-level quantum system driven by the four independent Ornstein-Uhlenbeck processes. The SSE has unique properties which are hard to achieve in other approaches to non-Markovian dynamics. Especially, it is the complete positivity of the density operator $\tilde\rho(t) = \mathbb{E}(|\tilde\psi(t)\rangle\langle\tilde\psi(t)|)$ for all time.

Another very interesting feature of the stochastic Schrödinger equation is its dimension. The dimension of SSE is the square root of the dimension of the corresponding master equation. Of course it opens new possibilities to study high dimensional open quantum systems even in non-Markovian regime. We feel that in the higher dimensions SSEs do not have an alternative. Quantum dynamics given by the non-Markovian SSE is significantly differ from Markovian one and this fact should be taken into account in the explanation of future experiments with quantum ensembles.

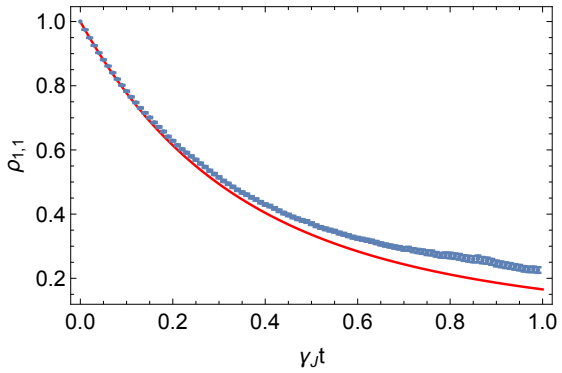

Fig.1. Evolution of the upper state. Red curve is the Markovian dynamics and blue dots are the non-Markovian dynamics. Parameters: $\gamma_K = 2\gamma_J$, $N_J = 0.2$, $N_K = 0.3$, $k_J^1 = k_K^1 = 0.3$, $k_J^2 = k_K^2 = 0.5$.

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