New many-parameter quaternion Fourier-Clifford-Hamilton transforms for intelligent OFDM TCS

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Abstract. In this paper, we aim to investigate the superiority and practicability of many-parameter quaternion Fourier transforms (MPQFT) from the physical layer security (PHY-LS) perspective. We propose novel Intelligent OFDM-telecommunication system (Intelligent-OFDM-TCS), based on MPQFT. New system uses inverse MPQFT for modulation at the transmitter and direct MPQFT for demodulation at the receiver. The purpose of employing the MPFTs is to improve the PHY-LS of wireless transmissions against to the wide-band anti-jamming communication. Each MPQFT depends on finite set of independent parameters (angles), which could be changed independently one from another. When parameters are changed, multi-parametric transform is also changed taking form of a set known (and unknown) orthogonal (or unitary) transforms.

1. Introduction

With most of the data transmission systems nowadays use orthogonal frequency division multiplexing telecommunication system (OFDM-TCS) based on the discrete Fourier transform (DFT). Some versions of it is: digital audio broadcast (DAB), digital video broadcast (DVB), and wireless local area network (WLAN), standards such as IEEE802.11g and long term evolution (LTE and its extension LTE-Advanced, Wi-Fi (IEEE 802.11), worldwide interoperability for microwave ACCESS (WiMAX IEEE 802.16) or ADSL [1]. The concept of using parallel data broadcast by means of frequency division multiplexing (FDM) was printed in mid 60s [2]. The conventional OFDM is a multi-carrier modulation technique that is basic technology having high-speed transmission capability with bandwidth efficiency and robust performance in multipath fading environments. OFDM divides the available spectrum into a number of parallel orthogonal sub-carriers and each sub-carrier is then modulated by a low rate data stream at different carrier frequency. In OFDM system, the modulation and demodulation can be applied easily by means of inverse and direct discrete Fourier transforms (DFT). The conventional OFDM will be denoted by the symbol F_N-**OFDM**. All sub-carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1}=\{e^{j2\pi kn/N}\}_{k=0}^{N-1}$ form matrix of discrete orthogonal Fourier transform $\mathrm{F}_N=[\mathbf{subc}_k(n)]_{k,n=0}^{N-1}\equiv\left[e^{j2\pi kn/N}\right]_{k,n=0}^{N-1}$. At the time, the idea of using the fast algorithm of different orthogonal transforms $\mathrm{U}_N=[\mathbf{subc}_k(n)]_{k,n=0}^{N-1}$ f or a software-based implementation of the OFDM's modulator and demodulator, transformed this technique from an attractive. OFDM-TCS, based on arbitrary orthogonal (unitary) transform U_N will be denoted as U_N-$\mathbf{OFDM}$. The idea which links F_N-$\mathbf{OFDM}$ and U_N-$\mathbf{OFDM}$ is that, in the same manner that the complex exponentials $\{e^{j2\pi kn/N}\}_{k=0}^{N-1}$ are orthogonal to each-other, the members of a family of U_N-sub-carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1}$ will satisfy the same property.

The U_N-$\mathbf{OFDM}$ reshapes the multi-carrier transmission concept, by using carriers $\{\mathbf{subc}_k(n)\}_{k=0}^{N-1}$ instead of OFDM's complex exponentials $\{e^{j2\pi kn/N}\}_{k=0}^{N-1}$. There are a number of candidates for orthogonal function sets used in the OFDM-TCS: discrete wavelet sub-carriers [3], Golay complementary sequences [4], Walsh functions [5], pseudo random sequences [6].

In this work, we propose a simple and effective anti-eavesdropping and anti-jamming Intelligent OFDM system, based on quaternion many-parameter transform (QMPT) $\mathrm{QU}_N(\varphi_1,\varphi_2,\dots,\varphi_q)$ instead of ordinary DFT F_N.

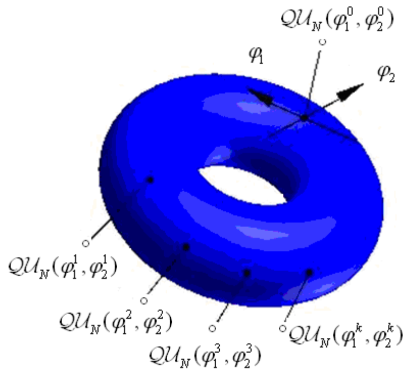

Figure 1. q-D torus $\mathbf{Tor}_q[0,2\pi]=[0,2\pi]^q$

Each QMPT depends on finite set of free parameters $\mathbf{\theta}=(\varphi_1,\varphi_2,\dots,\varphi_q)$, and each of them can take its value form 0 to 2π. When parameters are changed, QMPT is changed too taking form of known (and unknown) quaternion transforms. The vector of parameters $\mathbf{\theta}=(\varphi_1,\varphi_2,\dots,\varphi_q)\in\mathbf{Tor}_q[0,2\pi]=[0,2\pi]^q$ belongs to the q-D torus (see Fig. 1). When the vector of parameters $(\varphi_1,\varphi_2,\dots,\varphi_q)$ runs completely the q-D torus $\mathbf{Tor}_q[0,2\pi]$, the ensemble of the orthogonal quaternion transforms is created. Intelligent OFDM system uses some concrete values of the parameters $\varphi_1=\varphi_1^0,\varphi_2=\varphi_2^0,\dots,\varphi_q=\varphi_q^0$, *i.e.*, it uses a concrete realization of QMPT $\mathrm{QU}_N^0\equiv\mathrm{QU}_N(\varphi_1^0,\varphi_2^0,\dots,\varphi_q^0)$. The vector $(\varphi_1^0,\varphi_2^0,\dots,\varphi_q^0)$ plays the role of some analog key (see Fig. 2), whose knowing is necessary for entering into the TCS with the aim of intercepting the confidential information.

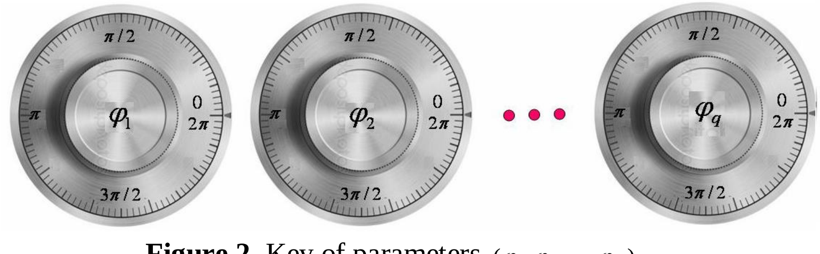

Figure 2. Key of parameters $(\varphi_1,\varphi_2,\dots,\varphi_q)$

Quantity of parameters can achieve the values $p=10\ 000$. So, searching the vector key by scanning the 10000-dimensional torus $[0,2\pi]^{10\ 000}$ with the aim of finding the working parameters

$(\varphi_1^0, \varphi_2^0, \ldots, \varphi_q^0)$ is very difficult problem for the enemy cyber-means. But if, nevertheless, this key were found by the enemy in the cyber attack, then the system could change values of the working parameters for rejecting the enemy attack. If the system is one of the TCP type, then in such a case, it will transmit the confidential information on the new sub-carriers (*i.e.*, in the new orthogonal basis). As a result, the system will counteract against the enemy radio-electronic attacks.

The enemy problem is also complicated by the fact that the QMPT is additionally arranged with digital key that is joined with the non-commutativity of the quaternion multiplication operation. In the QMPT matrix multiplication on the data vector, each element of the vector can be multiplied on the matrix element either from the left or from the right. Let the symbol "0" means multiplication of the data vector $\vec{\mathbf{v}} = (v_1, \ldots, v_k, \ldots, v_N)$ on the matrix element from the left (L) and the symbol "1" means the multiplication from the right (R). Then the N -D binary vector $(b_1, b_2, \ldots, b_N)$ is the digital key (see Fig.3) showing onto the way, by which the multiplication of the QMPT matrix on the data vector has to be implemented:

$$\left[\mathrm{qmp}_k^{b_k}\left(n \mid \theta_1, \theta_2, \ldots, \theta_p\right)\right] \cdot \vec{\mathbf{v}} \Rightarrow \begin{cases} \mathrm{qmp}_k^{b_k}(n) \cdot v_k, & b_k = 0, \\ v_k \cdot \mathrm{qmp}_k^{b_k}(n), & b_k = 1, \end{cases}.$$

where $\left\{\mathrm{qmp}_k^{b_k}(n)\right\}_{k,n=1}^{N}$ are the set of matrix elements of quaternion transform, *i.e.*,

$$\mathrm{QU}^{(b_1, b_2, \ldots, b_N)}(\varphi_1, \varphi_2, \ldots, \varphi_q) = \left[\mathrm{qmp}_k^{b_k}\left(n \mid \theta_1, \theta_2, \ldots, \theta_p\right)\right]_{k,n=1}^{N}.$$

So, the number of such keys is equal to 2^N. They form the Boolean cube $\mathbf{B}_2^N$. Knowing this key is necessary to enter into the Intelligent OFDM TCS.

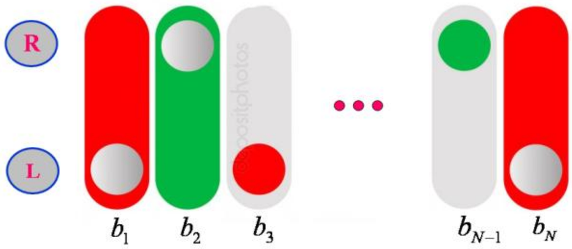

Figure 3. N -D binary vector-key $(b_1, b_2, \ldots, b_N)$

Thus, the space of keys is the Descartes product $\mathbf{Tor}_q[0, 2\pi] \times \mathbf{B}_2^N$. Searching the pair of keys $(\varphi_1, \varphi_2, \ldots, \varphi_q) \in \mathbf{Tor}^q[0, 2\pi]$ and $(b_1, b_2, \ldots, b_N) \in \mathbf{B}_2^N$ in the space of keys is very difficult problem for the enemy, especially, when we have the OFDM TCS. Such a system can also defend itself by changing the values of the working parameters and crypto key accordingly to some law that is known for the transmitter and receiver in advance. The law can be deterministic or the pseudo-random (similarly to the law, by which the contemporary TCSs change their working frequency). The main advantage of using QMPT in OFDM TCS is that it is a very flexible anti-eavesdropping and anti-jamming Intelligent OFDM TCS. These TCS have additional advantages in comparison with the classic TCS: the multi-parametric transforms allow one to optimize (and as a result, to enhance) the technical characteristics of the system (by changing its parameters) such as the PARP (peak to average power ratio), BER (bit error rate), SER (symbol error rate), and the ISI (inter-symbol interference).

The paper are organized as follows. Section 2 of the paper presents a brief introduction to the quaternion algebra. Section 3 and 4 present quaternion Fourier transforms

2. Quaternions

The space of quaternions denoted by $\mathrm{H}(\mathbf{R})$ were first invented by W.R. Hamilton in 1843 as an extension of the complex numbers into four dimensions [7]. General information on quaternions may be obtained from [8].

Definition 1. Numbers of the form ${}^4\mathbf{q} = a\mathbf{1} + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$, where $a,b,c,d \in \mathbf{R}$ are called quaternions, where 1) $\mathbf{1}$ is the real unit; 2) $\mathbf{i},\mathbf{j},\mathbf{k}$ are three imaginary units.

We speck that quaternions ${}^4\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ are written in the standard format. The addition and subtraction of two quaternions ${}^4\mathbf{q}_1 = a_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k}$ and ${}^4\mathbf{q}_2 = a_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k}$ are given by

$${}^4\mathbf{q}_1 \pm {}^4\mathbf{q}_2 = (a_1 \pm a_2) + (b_1 \pm b_2)\mathbf{i} + (b_1 \pm b_2)\mathbf{j} + (b_1 \pm b_2)k.$$

The product of quaternions for the standard format Hamilton defined according as:

$${}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2 = (a_1 + b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}) \circ (a_2 + b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}) = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) +$$
$$+ (a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)\mathbf{i} + (a_1c_2 + c_1a_2 + d_1b_2 - b_1d_2)\mathbf{j} + (a_1d_2 + d_1a_2 + b_1c_2 - c_1b_2)\mathbf{k},$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1;\ \mathbf{i} \circ \mathbf{j} = -\mathbf{i} \circ \mathbf{j} = \mathbf{k},\ \mathbf{i} \circ \mathbf{k} = -\mathbf{k} \circ \mathbf{i} = \mathbf{j},\ \mathbf{j} \circ \mathbf{k} = -\mathbf{k} \circ \mathbf{j} = \mathbf{i}$. The set of quaternions with operations of multiplication and addition forms 4-D algebra $\mathrm{H}(\mathbf{R}) = \mathrm{H}(\mathbf{R} \mid 1, \mathbf{i}, \mathbf{j}, \mathbf{k}) := \mathbf{R} + \mathbf{R}\mathbf{i} + \mathbf{R}\mathbf{j} + \mathbf{R}\mathbf{k}$ over the real field $\mathbf{R}$.

Number component a and direction component ${}^3\mathbf{r} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbf{R}^3$ were called the *scalar* and 3-D *vector* parts of quaternion, respectively. Now these components are denoted as $S({}^4\mathbf{q}) = a \in \mathbf{R}$ and $V(\mathbf{q}) = {}^3\mathbf{r} = b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$. A non–zero element ${}^3\mathbf{r} = bi + cj + dk$ is called pure vector quaternion. Hence, according to W. Hamilton every quaternion is the sum of a scalar number and a pure vector quaternion ${}^4\mathbf{q} = a + (b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) = a + {}^3\mathbf{r} = S(\mathbf{q}) + V(\mathbf{q})$, where $a = S({}^4\mathbf{q})$, $V({}^4\mathbf{q}) = {}^3\mathbf{r}$. Since $\mathbf{i} \circ \mathbf{j} = \mathbf{k}$, then a quaternion ${}^4\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = (a1 + b\mathbf{i}) + (c\mathbf{j} + d\mathbf{i} \circ \mathbf{j}) = (a + b\mathbf{i}) + (c + d\mathbf{i}) \circ \mathbf{j} = \mathbf{z} + \mathbf{w} \circ \mathbf{j}$ is the sum of two complex numbers $\mathbf{z} = a + b\mathbf{i}$, $\mathbf{w} = c + d\mathbf{i}$ with a new imaginary unit $\mathbf{j}$. So, every quaternion can be represented in several ways:

(1) as a 4-D hypercomplex number ${}^4\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} = (a,b,c,d)$, $a,b,c,d \in \mathbf{R}$ (standard 4-D format);

(2) as a sum of a scalar and vector parts $\mathbf{q} = a + {}^3\mathbf{r} = (a, {}^3\mathbf{r})$ (1,3-D hypercomplex format);

(3) as a 2-D hypercomplex numbers ${}^{2,2}\mathbf{q} = \mathbf{z} + \mathbf{w} \circ \mathbf{j} = (\mathbf{z}, \mathbf{w})$, $\mathbf{z}, \mathbf{w} \in \mathbf{C}$ (2,2-D complex format).

The product of quaternions for the last two forms Hamilton defined as:

$${}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2 = (\mathbf{z}_1 + \mathbf{w}_1 \circ \mathbf{j}) \circ (\mathbf{z}_2 + \mathbf{w}_2 \circ \mathbf{j}) = (\mathbf{z}_1\mathbf{z}_2 - \mathbf{w}_1\overline{\mathbf{w}}_2) + (\mathbf{w}_1\overline{\mathbf{z}}_2 + \mathbf{z}_1\mathbf{w}_2) \circ \mathbf{j},$$
$${}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2 = (a_1 + \vec{\mathbf{r}}_1) \circ (a_2 + \vec{\mathbf{r}}_2) = \left(a_1a_2 - (\vec{\mathbf{r}}_1, \vec{\mathbf{r}}_2)\right) + \left(a_1\vec{\mathbf{r}}_2 + a_2\vec{\mathbf{r}}_1 + [\vec{\mathbf{r}}_1 \times \vec{\mathbf{r}}_2]\right),$$

where $S({}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2) = a_1a_2 - \left\langle {}^3\mathbf{r}_1 \mid {}^3\mathbf{r}_2 \right\rangle$, $V({}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2) = a_1{}^3\mathbf{r}_2 + a_2{}^3\mathbf{r}_1 + {}^3\mathbf{r}_1 \times {}^3\mathbf{r}_2$. Here $\left\langle {}^3\mathbf{r}_1 \mid {}^3\mathbf{r}_2 \right\rangle = b_1b_2 + c_1c_2 + d_1d_2$ and ${}^3\mathbf{r}_1 \times {}^3\mathbf{r}_2 = \mathbf{i}(c_1d_2 - d_1c_2) - \mathbf{j}(b_1d_2 - d_1b_2) + \mathbf{k}(b_1c_2 - c_1b_2)$ are scalar and vector products, respectively.

Definition 2. Let ${}^4\mathbf{q} = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathrm{H}(\mathbf{R})$ be a quaternion ($a,b,c,d \in \mathbf{R}$). Then

$${}^4\overline{\mathbf{q}} = \overline{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}} = a - b\mathbf{i} - c\mathbf{j} - d\mathbf{k} \quad \left({}^4\overline{\mathbf{q}} = \overline{a + {}^3\mathbf{r}} = a - {}^3\mathbf{r}\right)$$

is the conjugate of ${}^4\mathbf{q}$, $N({}^4\mathbf{q}) = \|{}^4\mathbf{q}\| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{{}^4\overline{\mathbf{q}} \circ {}^4\mathbf{q}} = \sqrt{{}^4\mathbf{q} \circ {}^4\overline{\mathbf{q}}}$ is the norm of ${}^4\mathbf{q}$, and $tr({}^4\mathbf{q}) = 2a = {}^4\mathbf{q} + {}^4\overline{\mathbf{q}}$ is the trace of ${}^4\mathbf{q}$. Therefore ${}^4\mathbf{q}^2 - tr({}^4\mathbf{q}){}^4\mathbf{q} + N^2({}^4\mathbf{q}) = 0$.

Proposition 1. We have $\overline{{}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2} = {}^4\overline{\mathbf{q}}_2 \circ {}^4\overline{\mathbf{q}}_1$ and $N({}^4\mathbf{q}_1 \circ {}^4\mathbf{q}_2) = N({}^4\mathbf{q}_1) \cdot N({}^4\mathbf{q}_2)$ for every ${}^4\mathbf{q}_1, {}^4\mathbf{q}_2 \in \mathrm{H}(\mathbf{R})$. Note that $\|1\| = 1$, $\|\mathbf{i}\| = \|\mathbf{j}\| = \|\mathbf{k}\| = 1$.

Definition 3. Quaternions $\left\{ {}^4\boldsymbol{\rho} \middle| N({}^4\boldsymbol{\rho}) = 1 \right\}$ of unit norm are called unit quaternions.

The unit quaternions $\boldsymbol{\rho}$ form a 3D hypersphere $\mathrm{S}^3 \subset \mathrm{H}(\mathbf{R}) \cong \mathbf{R}^4$. For each quaternion ${}^4\mathbf{q}$ with nonzero norm the following quaternion

$${}^4\boldsymbol{\rho} = \frac{{}^4\mathbf{q}}{\|{}^4\mathbf{q}\|} = \frac{a + {}^3\mathbf{r}}{\|{}^4\mathbf{q}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{{}^3\mathbf{r}}{\|{}^4\mathbf{q}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{\|{}^3\mathbf{r}\|}{\|{}^4\mathbf{q}\|}\frac{{}^3\mathbf{r}}{\|{}^3\mathbf{r}\|} = \frac{a}{\|{}^4\mathbf{q}\|} + \frac{\|{}^3\mathbf{r}\|}{\|{}^4\mathbf{q}\|}{}^3\boldsymbol{\mu} =$$
$$= \cos\alpha + {}^3\boldsymbol{\mu}\sin\alpha = \cos\alpha + (\mu_1\mathbf{i} + \mu_2\mathbf{j} + \mu_3\mathbf{k})\sin\alpha$$

is an unit quaternion, where

$$\left\|{}^3\mathbf{r}\right\| = \sqrt{b^2+c^2+d^2},\ {}^3\boldsymbol{\mu} = {}^3\mathbf{r}/\left\|{}^3\mathbf{r}\right\|,\ \cos\alpha = a/\left\|{}^4\mathbf{q}\right\|,\quad \sin\alpha = \left\|{}^3\mathbf{r}\right\|/\left\|{}^4\mathbf{q}\right\|,\ \mu_1 = b/\left\|{}^3\mathbf{r}\right\|,\ \mu_2 = c/\left\|{}^3\mathbf{r}\right\|,\ \mu_3 = d/\left\|{}^3\mathbf{r}\right\|$$

and $\boldsymbol{\mu} = \mu_1\mathbf{i} + \mu_2\mathbf{j} + \mu_3\mathbf{k}$. Obviously,

$${}^4\mathbf{q} = \left\|{}^4\mathbf{q}\right\|\cdot\left[\cos\alpha + {}^3\boldsymbol{\mu}(\gamma,\theta)\sin\alpha\right] = \left\|{}^4\mathbf{q}\right\|\cdot\left[\cos\alpha + (\mu_1\mathbf{i} + \mu_2\mathbf{j} + \mu_3\mathbf{k})\sin\alpha\right].$$

where ${}^3\boldsymbol{\mu}(\gamma,\theta) = \mathbf{i}\cos\gamma + \mathbf{j}\sin\gamma\cos\theta + \mathbf{k}\sin\gamma\sin\theta \in \mathbf{S}^2$, $\theta,\varphi \in [0,\pi]$, $\alpha \in [0,2\pi)$ are the polar coordinates on S^3. Obviously,

$$a = \left\|{}^4\mathbf{q}\right\|\cos\alpha,\ b = \left(\left\|{}^4\mathbf{q}\right\|\cos\gamma\right)\sin\alpha,\quad c = \left(\left\|{}^4\mathbf{q}\right\|\sin\gamma\cos\theta\right)\sin\alpha,\ d = \left(\left\|{}^4\mathbf{q}\right\|\sin\gamma\sin\theta\right)\sin\alpha.$$

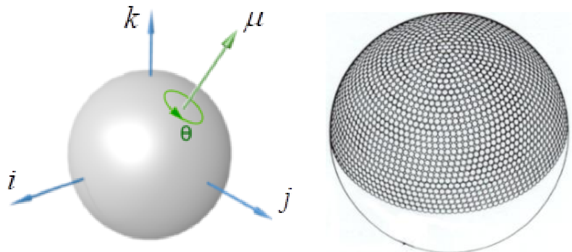

Figure 4. Each 3D vector $\vec{\boldsymbol{\mu}} \in S^2$ of unit length can play a role of classical imaginary unit. For example, the special elements ${}^3\mathbf{i}, {}^3\mathbf{j}, {}^3\mathbf{k}$ are such elements.

In particular, for ${}^4\mathbf{q}_1 \equiv {}^3\mathbf{r}_1 = b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$ and ${}^4\mathbf{q}_2 \equiv {}^3\mathbf{r}_2 = b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}_2$ we obtain

$${}^3\mathbf{r}_1 \circ {}^3\mathbf{r}_2 = -\left\langle {}^3\mathbf{r}_1) \mid {}^3\mathbf{r}_2 \right\rangle + \left[{}^3\mathbf{r}_1 \times {}^3\mathbf{r}_2)\right],\quad {}^3\mathbf{r}^2 = {}^3\mathbf{r} \circ {}^3\mathbf{r} = -\left\langle {}^3\mathbf{r} \mid {}^3\mathbf{r} \right\rangle = -\left\|{}^3\mathbf{r}\right\|^2,$$

and for a pure quaternion ${}^3\boldsymbol{\mu} \in S^2 \subset \mathbf{R}^3$ with unity norm $\left\|{}^3\boldsymbol{\mu}\right\| = 1$ we have ${}^3\boldsymbol{\mu}^2 = -\left\|{}^3\boldsymbol{\mu}\right\|^2 = -1$, where S^2 denotes the unit 2-D sphere in 3-D space $\mathbf{R}^3$. This unit-vector product identity represents the generalization of the complex-variable identity $i^2 = -1$.

This means that, if in the ordinary theory of complex numbers there are only two different square roots of negative unity ($+i$ and $-i$) and they differ only in their signs, then in the quaternion theory there are infinite numbers of different square roots of negative unity

$${}^3\boldsymbol{\mu} = {}^3\boldsymbol{\mu}(\gamma,\theta) = \left(\mu_x\mathbf{i} + \mu_y\mathbf{j} + \mu_z\mathbf{k}\right) = \left(\cos\gamma\cdot\mathbf{i} + \sin\gamma\cos\theta\cdot\mathbf{j} + \sin\gamma\sin\theta\cdot\mathbf{k}\right) \in \mathbf{S}^2,$$

which gives ${}^3\boldsymbol{\mu}^2 = {}^3\boldsymbol{\mu}^2(\gamma,\theta) = -1$. Here ${}^3\boldsymbol{\mu}(\gamma,\theta) = (\cos\gamma, \sin\gamma\cos\theta, \sin\gamma\sin\theta)$ being still that point on the spherical surface, which has for its rectangular coordinates $\cos\gamma, \sin\gamma\cos\theta, \sin\gamma\sin\theta$ (see Fig. 4). In the feature we will omit left index: $\boldsymbol{\mu}(\gamma,\theta) \equiv {}^3\boldsymbol{\mu}(\gamma,\theta)$.

Definition 4. A functions ${}^4\mathbf{f}(n) : [0, N-1] \to \mathrm{H}(\mathbf{R})$ are called quaternion-valued discrete functions. They have the following form:

$${}^4\mathbf{f}(n) = f_0(n) + f_1(n)\mathbf{i} + f_2(n)\mathbf{j} + f_3(n)\mathbf{k} = \left(f_0(n), f_1(n), f_2(n), f_3(n)\right).$$

The exponential function is $\exp({}^4\mathbf{q}) = 1 + {}^4\mathbf{q} + \frac{{}^4\mathbf{q}^2}{2!} + ... + \frac{{}^4\mathbf{q}^m}{m!} + ... = \sum_{m=0}^{\infty}\frac{{}^4\mathbf{q}^m}{m!}$.

Theorem 1 [8]. For ${}^4\mathbf{q} = a + {}^3\mathbf{r} \in \mathrm{H}(\mathbf{R})$ we have

$$\exp(a + {}^3\mathbf{r}) = e^a\exp({}^3\mathbf{r}) = e^a\left(\cos(\|{}^3\mathbf{r}\|) + \frac{{}^3\mathbf{r}}{\|{}^3\mathbf{r}\|}\sin(\|{}^3\mathbf{r}\|)\right).$$

Obviously, $\|\exp({}^3\mathbf{r})\| = 1$ and $\|\exp({}^3\mathbf{r})\| = \|\exp(a + {}^3\mathbf{r})\| = e^a$. In general case

$$\exp({}^4\mathbf{q}_1) \circ \exp({}^4\mathbf{q}_2) \neq \exp({}^4\mathbf{q}_2) \circ \exp({}^4\mathbf{q}_1)$$

and

$$\exp({}^4\mathbf{q}_1 + {}^4\mathbf{q}_2) \neq \exp({}^4\mathbf{q}_1) \circ \exp({}^4\mathbf{q}_2) \neq \exp({}^4\mathbf{q}_2) \circ \exp({}^4\mathbf{q}_1).$$

3. Quaternion Fourier transforms

Before defining the quaternion Fourier transform, we briefly outline its relationship with Clifford Fourier transformations. Quaternions and Clifford hypercomplex number were first simultaneously and independently applied to quaternion-valued Fourier and Clifford-valued Fourier transforms by Labunets [9] and Sommen [10,11], respectively, at the 1981. The Labunets quaternion transforms were over quaternion with real and Galois coefficients (i.e., over $\mathrm{H}[\mathbf{R}]$ and $\mathrm{H}[\mathbf{GF}(p)]$). They generalize both classical and co-called number theoretical transforms (NNTs) and proposed for application to fast signal processing. Ernst [12] and Delsuc [13] in the late 1981s, seemingly without knowledge of the earlier works of Labunets and Sommen proposed bicomplex Fourier transforms over 4D commutative hypercomplex algebra of bicomplex numbers ($\mathbf{C} \oplus \mathbf{C}$). Note that the bicomplex algebra is quite different from the quaternion algebra; among general things, bicomplex multiplication is commutative, but quaternion one is noncommutative. For this reason, the Ernst and Delsuc transforms are direct sum of ordinary Fourier transforms (*i.e.*, duplex Fourier transform). They are a little bit similar in kind to quaternion Fourier transforms. Ernst and Delsuc's transforms were two-dimensional and proposed for application to nuclear magnetic resonance (NMR) imaging.

Two new ideas emerged in 1998-1999 in a paper by Labunets [14] and Sangwine [15]. These were, firstly, the choice of a general root ${}^3\boldsymbol{\mu}$ of -1 (a unit quaternion with zero scalar part) rather than a basis unit ($\mathbf{i}, \mathbf{j}$ or $\mathbf{k}$) of the quaternion algebra, and secondly, the choice of a general roots ${}^3\boldsymbol{\mu}_0 = \boldsymbol{\mu}_0(\gamma_0, \theta_0), {}^3\boldsymbol{\mu}_1 = \boldsymbol{\mu}_1(\gamma_1, \theta_1), \ldots, {}^3\boldsymbol{\mu}_{N-1} = \boldsymbol{\mu}_{N-1}(\gamma_{N-1}, \theta_{N-1})$ of -1 (see cloud of imaginary units on Fig.1) in Clifford algebra to create multi-parameter and fractional Fourier-Clifford transforms (with eigenvalues $e^{-\boldsymbol{\mu}_0(\gamma_0,\theta_0)}, e^{-\boldsymbol{\mu}_1(\gamma_1,\theta_1)}, \ldots, e^{-{}^3\boldsymbol{\mu}_{N-1}(\varphi_{N-1},\theta_{N-1})}, \ldots$

Labunets, Rundblad-Ostheimer and Astola [16-18] used the classical and number theoretical quaternion Fourier and Fourier-Clifford transforms for fast invariant recognition of 2D, 3D and nD color and hyperspectral images, defined on Euclidean and non-Euclidean spaces. These publications give useful interpretation of quaternion and Cliffordean Fourier coefficients: they are relative quaternion- or Clifford-valued invariants of hyperspectral images with respect to Euclidean and non-Euclidean rotations and motions of physical and hyperspectral spaces. It removes the veil of mysticism and mystery from quaternion- and Clifford-valued Fourier coefficients. In the works of scientists F.Brackx, H. De Schepper, F. Sommen, and H. De Bie [19-22] mathematical theory of Fourier-Clifford transforms accepted the final completeness, beauty and elegance.

According to Theorem 1, for non-zero $\alpha \in \mathbf{R}$ and a non-zero ${}^4\mathbf{q} = a + {}^3\boldsymbol{\mu}$

$$\exp({}^4\mathbf{q}\alpha) = \exp\left((a + {}^3\boldsymbol{\mu})\alpha\right) = e^{a\alpha}\left(\cos(\|{}^3\boldsymbol{\mu}\|\alpha) + \frac{{}^3\boldsymbol{\mu}}{\|{}^3\boldsymbol{\mu}\|}\sin(\|{}^3\boldsymbol{\mu}\|\alpha)\right).$$

In particular case, for ${}^4\mathbf{q} \equiv {}^3\boldsymbol{\mu} = \boldsymbol{\mu}(\gamma, \theta)$ we have $e^{\boldsymbol{\mu}(\gamma,\theta)\alpha} = \cos(\alpha) + \boldsymbol{\mu}(\gamma,\theta)\sin(\alpha)$. For $\alpha = \alpha_k = 2\pi k / N$ $(k = 0,1,\ldots, N-1)$ we obtain quaternion-valued discrete harmonics

$$e^{\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn} = \boldsymbol{\varepsilon}_k^{kn} = \cos\left(\frac{2\pi}{N}kn\right) + \boldsymbol{\mu}(\gamma_k,\theta_k)\sin\left(\frac{2\pi}{N}kn\right),$$

where each quaternion harmonic $\boldsymbol{\varepsilon}_k^{-kn} = \exp\left(-2\pi\boldsymbol{\mu}(\gamma_k,\theta_k)kn/N\right)$ has its own imaginary unit $\boldsymbol{\mu}_k := \boldsymbol{\mu}(\gamma_k,\theta_k) = \left(\cos\gamma_k \cdot \mathbf{i} + \sin\gamma_k\cos\theta_k \cdot \mathbf{j} + \sin\gamma_k \sin\theta_k \cdot \mathbf{k}\right) \in \mathbf{S}^2$, $k = 0,1,\ldots, N-1$. Due to the non-commutative property of quaternion multiplication, there are two different types of quaternion Fourier transforms (QFTs). These QFTs are the left- and right-sided QFTs (LS-QFT and RS-QFT), respectively.

Definition 5. The direct discrete quaternion Fourier transforms of $\mathbf{f}(n):[0, N-1] \to \mathrm{H}(\mathbf{R})$ are defined as

$${}^4\mathbf{QF}\left(k\,\middle|\,\gamma_k,\theta_k\right) = \mathrm{QF}^{(\gamma,\boldsymbol{\theta})}\left\{{}^4\mathbf{f}(n)\right\} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{-\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn} \circ {}^4\mathbf{f}(n) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}\boldsymbol{\varepsilon}_k^{-kn} \circ {}^4\mathbf{f}(n), \quad (1)$$

$$ {}^4\mathbf{FQ}\left(k\left|\varphi_k,\theta_k\right.\right) = \mathcal{F}Q^{(\boldsymbol{\varphi},\boldsymbol{\theta})}\left\{{}^4\mathbf{f}(n)\right\} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}{}^4\mathbf{f}(n)\circ e^{-\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}{}^4\mathbf{f}(n)\circ\boldsymbol{\varepsilon}_k^{-kn}, \qquad (2) $$

where $Q\mathcal{F}^{(\boldsymbol{\gamma},\boldsymbol{\theta})}$, $\mathcal{F}Q^{(\boldsymbol{\gamma},\boldsymbol{\theta})}$ are LS-QFT and RS-QFT, $\boldsymbol{\gamma} = (\gamma_0,\gamma_1,\ldots,\gamma_{N-1})$, $\boldsymbol{\theta} = (\theta_0,\theta_1,\ldots,\theta_{N-1})$.

Definition 6. The inverse quaternion Fourier transforms are defined as

$$ {}^4\mathbf{f}(n) = 2\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\frac{\boldsymbol{\mu}(\gamma_k,\theta_k)}{\left[\boldsymbol{\mu}(\gamma_{-k},\theta_{-k})+\boldsymbol{\mu}(\gamma_k,\theta_k)\right]_L}\circ\boldsymbol{\varepsilon}_k^{kn}\circ{}^4\mathbf{QF}\left(k\left|\gamma_k,\theta_k\right.\right), \qquad (3) $$

$$ {}^4\mathbf{f}(n) = 2\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}{}^4\mathbf{QF}\left(k\left|\gamma_k,\theta_k\right.\right)\circ\boldsymbol{\varepsilon}_k^{kn}\circ\frac{\boldsymbol{\mu}(\gamma_k,\theta_k)}{\left[\boldsymbol{\mu}(\gamma_{-k},\theta_{-k})+\boldsymbol{\mu}(\gamma_k,\theta_k)\right]_R}. \qquad (4) $$

We see, that $Q\mathcal{F} \equiv Q\mathcal{F}^{(\boldsymbol{\gamma},\boldsymbol{\theta})}$ and $\mathcal{F}Q \equiv \mathcal{F}Q^{(\boldsymbol{\gamma},\boldsymbol{\theta})}$ depend on $2N$ parameters (γ_k,θ_k), $k\in\{0,1,\ldots,N-1\}$.

It will be convenient to think of these parameters as $2N$ -dimension vector $(\gamma_0,\theta_0;\gamma_1,\theta_1;\ldots;\gamma_{N-1},\theta_{N-1})$.

4. Demystification of the Fourier-Clifford Transforms

It is an interesting problem to study the interrelation between the classical and Fourier-Clifford transforms. We need in the classical 4-channel Fourier transform (quadruple Fourier transform)

$$ \left|{}^4\mathbf{F}(n)\right\rangle = \mathcal{F}\left|{}^4\mathbf{f}(n)\right\rangle = \mathcal{F}\left|f_0(n)\right\rangle + \mathcal{F}\left|f_1(n)\right\rangle\mathbf{i} + \mathcal{F}\left|f_2(n)\right\rangle\mathbf{j} + \mathcal{F}\left|f_3(n)\right\rangle\mathbf{k} \equiv $$

$$ = \left|F_0(k)\right\rangle + \left|F_1(k)\right\rangle\mathbf{i} + \left|F_2(k)\right\rangle\mathbf{j} + \left|F_3(k)\right\rangle\mathbf{k} \equiv $$

$$ = \left|\mathrm{R}_0(k) - i\mathrm{J}_0(k)\right\rangle + \left|\mathrm{R}_1(k) - i\mathrm{J}_1(k)\right\rangle\mathbf{i} + \left|\mathrm{R}_2(k) - i\mathrm{J}_2(k)\right\rangle\mathbf{j} + \left|\mathrm{R}_3(k) - i\mathrm{J}_3(k)\right\rangle\mathbf{k} \equiv $$

$$ = \left|\mathrm{R}_0(k) + \mathrm{R}_1(k)\mathbf{i} + \mathrm{R}_2(k)\mathbf{j} + \mathrm{R}_3(k)\mathbf{k}\right\rangle - i\left|\mathrm{J}_0(k) + \mathrm{J}_1(k)\mathbf{i} + \mathrm{J}_2(k)\mathbf{j} + \mathrm{J}_3(k)\mathbf{k}\right\rangle \equiv $$

$$ = \left|{}^4\mathbf{R}(k)\right\rangle + i\left|{}^4\mathbf{J}(k)\right\rangle = \begin{bmatrix} {}^4\mathbf{R}(0) \\ {}^4\mathbf{R}(1) \\ \cdots \\ {}^4\mathbf{R}(N-1) \end{bmatrix} - i\begin{bmatrix} {}^4\mathbf{J}(0) \\ {}^4\mathbf{J}(1) \\ \cdots \\ {}^4\mathbf{J}(N-1) \end{bmatrix} = \begin{bmatrix} {}^4\mathbf{F}(0) \\ {}^4\mathbf{F}(1) \\ \\ {}^4\mathbf{F}(N-1) \end{bmatrix}, $$

where $\mathrm{R}_m(k)$ and $\mathrm{J}_m(k)$ are real and imaginary parts of ordinary complex-valued Fourier spectra $\mathrm{F}_m(k) = \mathrm{R}_m(k) + i\mathrm{J}_m(k)$, *i.e.*,

$$ \mathrm{R}_m(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} f_m(n)\cos\frac{2\pi}{N}kn, \quad \mathrm{J}_m(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} f_m(n)\sin\frac{2\pi}{N}kn $$

for $m = 0,1,2,3$. Note, that for real-valued signals $f_m(n)$, we have $\mathrm{F}_m^*(k) = \mathrm{R}_m(k) - i\mathrm{J}_m(k) = \mathrm{F}_m(-k) = \mathrm{R}_m(-k) + i\mathrm{J}_m(-k)$, *i.e.*, $\mathrm{R}_m(k) = \mathrm{R}_m(-k)$ and $\mathrm{J}_m(k) = -\mathrm{J}_m(-k)$.

Theorem 2. If ${}^4\mathbf{F}(k) = {}^4\mathbf{R}(k) - i\cdot{}^4\mathbf{J}(k)$ are classical Fourier spectra, then quaternion Fourier spectra have the following forms

$$ {}^4\mathbf{QF}\left(k\left|\gamma_k,\theta_k\right.\right) = {}^4\mathbf{R}(k) - \boldsymbol{\mu}(\gamma_k,\theta_k)\circ{}^4\mathbf{J}(k), \quad {}^4\mathbf{FQ}\left(k\left|\gamma_k,\theta_k\right.\right) = {}^4\mathbf{R}(k) - {}^4\mathbf{J}(k)\circ\boldsymbol{\mu}(\gamma_k,\theta_k). \qquad (5) $$

where $\boldsymbol{\mu}(\gamma_k,\theta_k) = \left(\cos\gamma_k\cdot\mathbf{i} + \sin\gamma_k\cos\theta_k\cdot\mathbf{j} + \sin\gamma_k\sin\theta_k\cdot\mathbf{k}\right)$.

Proof. Indeed, for $\mathbf{QF}\left(k\left|\gamma_k,\theta_k\right.\right)$ we have

$$ {}^4\mathbf{QF}\left(k\left|\gamma_k,\theta_k\right.\right) = Q\mathcal{F}^{(\boldsymbol{\gamma},\boldsymbol{\theta})}\left\{{}^4\mathbf{f}(n)\right\} = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} e^{-\boldsymbol{\mu}(\gamma_k,\theta_k)\frac{2\pi}{N}kn}\circ{}^4\mathbf{f}(n) = $$

$$ = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}{}^4\mathbf{f}(n)\cos\left(\frac{2\pi}{N}kn\right) - \boldsymbol{\mu}(\gamma_k,\theta_k)\circ\frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}{}^4\mathbf{f}(n)\sin\left(\frac{2\pi}{N}kn\right) = $$

$$ = {}^4\mathbf{R}(k) - \boldsymbol{\mu}(\gamma_k,\theta_k)\circ{}^4\mathbf{J}(k), $$

where

$$ {}^4\mathbf{R}(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} {}^4\mathbf{f}(n)\cos\left(\frac{2\pi}{N}kn\right), \quad {}^4\mathbf{J}(k) = \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} {}^4\mathbf{f}(n)\sin\left(\frac{2\pi}{N}kn\right) $$

Analogously, ${}^4\mathbf{FQ}\left(k\,\middle|\gamma_k,\theta_k\right) = {}^4\mathbf{R}(k) - {}^4\mathbf{J}(k)\circ\boldsymbol{\mu}(\gamma_k,\theta_k)$.

So, if in classical Fourier theory the relation between real and imaginary parts is given by ${}^4\mathbf{F}(k) = {}^4\mathbf{R}(k) + i\cdot{}^4\mathbf{J}(k)$ then the quaternion-valued spectra are expressed through classical one as in (5), or in vector-matrix form as

$$ {}^4\overrightarrow{\mathbf{QF}}(k\mid\gamma_k,\theta_k) = {}^4\vec{\mathbf{R}}(k) - \hat{\boldsymbol{\mu}}^{LS}(\gamma_k,\theta_k)\cdot{}^4\vec{\mathbf{J}}(k), \quad {}^4\overleftarrow{\mathbf{QF}}(k\mid\gamma_k,\theta_k) = {}^4\vec{\mathbf{R}}(k) - \hat{\boldsymbol{\mu}}^{RS}(\gamma_k,\theta_k)\cdot{}^4\vec{\mathbf{J}}(k), \qquad (6) $$

where

$$ \hat{\boldsymbol{\mu}}^{LS}(\varphi_k,\theta_k) = \begin{bmatrix} \cdot & \cos\varphi_k & \sin\varphi_k\cos\theta_k & \sin\varphi_k\sin\theta_k \\ -\cos\varphi_k & \cdot & \sin\varphi_k\sin\theta_k & -\sin\varphi_k\cos\theta_k \\ -\sin\varphi_k\cos\theta_k & -\sin\varphi_k\sin\theta_k & \cdot & \cos\varphi_k \\ -\sin\varphi_k\sin\theta_k & \sin\varphi_k\cos\theta_k & -\cos\varphi_k & \cdot \end{bmatrix}, $$

$$ \hat{\boldsymbol{\mu}}^{RS}(\varphi_k,\theta_k) = \begin{bmatrix} \cdot & -\cos\varphi_k & -\sin\varphi_k\cos\theta_k & -\sin\varphi_k\sin\theta_k \\ \cos\varphi_k & \cdot & \sin\varphi_k\sin\theta_k & -\sin\varphi_k\cos\theta_k \\ \sin\varphi_k\cos\theta_k & -\sin\varphi_k\sin\theta_k & \cdot & \cos\varphi_k \\ \sin\varphi_k\sin\theta_k & \sin\varphi_k\cos\theta_k & -\cos\varphi_k & \cdot \end{bmatrix}. $$

Example 1. Let

$$ 1)\ \boldsymbol{\mu}(\varphi,\theta)\equiv\mathbf{i} = \overleftarrow{\boldsymbol{\mu}}(0,\pi/2) = (1,0,0),\quad 2)\ \boldsymbol{\mu}(\varphi,\theta)\equiv\mathbf{j} = \overleftarrow{\boldsymbol{\mu}}(\pi/2,0) = (0,1,0), $$

$$ 3)\ \boldsymbol{\mu}(\varphi,\theta)\equiv\mathbf{k} = \overleftarrow{\boldsymbol{\mu}}(\pi/2,\pi/2) = (0,1,0),\quad 4)\ \boldsymbol{\mu}\equiv(\mathbf{i}+\mathbf{j}+\mathbf{k})/\sqrt{3} = \left(\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right). $$

Then

$$ 1)\ \hat{\mathbf{i}}^{LS} = \begin{bmatrix} \cdot & -1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix},\ \hat{\mathbf{i}}^{RS} = \begin{bmatrix} \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & 1 & \cdot \end{bmatrix},\ 2)\ \hat{\mathbf{j}}^{LS} = \begin{bmatrix} \cdot & \cdot & -1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot \end{bmatrix},\ \hat{\mathbf{j}}^{RS} = \begin{bmatrix} \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & \cdot & 1 \\ -1 & \cdot & \cdot & \cdot \\ \cdot & \cdot-1 & \cdot & \cdot \end{bmatrix}, $$

$$ 3)\ \hat{\mathbf{k}}^{LS} = \begin{bmatrix} \cdot & \cdot & \cdot & -1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot \end{bmatrix},\ \hat{\mathbf{k}}^{RS} = \begin{bmatrix} \cdot & \cdot & \cdot & 1 \\ \cdot & \cdot & -1 & \cdot \\ \cdot & 1 & \cdot & \cdot \\ -1 & \cdot & \cdot & \cdot \end{bmatrix},\ 4)\ \hat{\boldsymbol{\mu}}^{LS} = \frac{\sqrt{3}}{3}\begin{bmatrix} \cdot & -1 & -1 & -1 \\ 1 & \cdot & -1 & 1 \\ 1 & 1 & \cdot & -1 \\ 1 & -1 & 1 & \cdot \end{bmatrix},\ \hat{\boldsymbol{\mu}}^{RS} = \frac{\sqrt{3}}{3}\begin{bmatrix} \cdot & 1 & 1 & 1 \\ -1 & \cdot & -1 & 1 \\ -1 & 1 & \cdot & -1 \\ -1 & -1 & 1 & \cdot \end{bmatrix}. $$

Hence,

$$ 1)\ {}^4\overrightarrow{\mathbf{QF}}(k\mid\mathbf{i}) = {}^4\vec{\mathbf{R}}(k) - \hat{\mathbf{i}}^{LS}\cdot{}^4\vec{\mathbf{J}}(k) = $$

$$ = \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{i}}^{LS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k)+\mathrm{J}_1(k) \\ \mathrm{R}_1(k)-\mathrm{J}_0(k) \\ \mathrm{R}_2(k)+\mathrm{J}_3(k) \\ \mathrm{R}_3(k)-\mathrm{J}_2(k) \end{bmatrix}, $$

$$ {}^4\overrightarrow{\mathbf{FQ}}(k\mid\mathbf{i}) = {}^4\vec{\mathbf{R}}(k) - \hat{\mathbf{i}}^{RS}\cdot{}^4\vec{\mathbf{J}}(k) = $$

$$ = \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{i}}^{RS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k)-\mathrm{J}_1(k) \\ \mathrm{R}_1(k)+\mathrm{J}_0(k) \\ \mathrm{R}_2(k)-\mathrm{J}_3(k) \\ \mathrm{R}_3(k)+\mathrm{J}_2(k \end{bmatrix}; $$

$$2)\quad {}^{4}\overrightarrow{\mathbf{QF}}(k\mid\mathbf{j}) = {}^{4}\vec{\mathbf{R}}(k) - \hat{\mathbf{j}}^{LS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{j}}^{LS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) + \mathrm{J}_2(k) \\ \mathrm{R}_1(k) - \mathrm{J}_3(k) \\ \mathrm{R}_2(k) - \mathrm{J}_0(k) \\ \mathrm{R}_3(k) + \mathrm{J}_1(k) \end{bmatrix},$$

$${}^{4}\overrightarrow{\mathbf{FQ}}(k\mid\mathbf{j}) = {}^{4}\vec{\mathbf{R}}(k) - \hat{\mathbf{j}}^{RS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{j}}^{RS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) - \mathrm{J}_2(k) \\ \mathrm{R}_1(k) - \mathrm{J}_3(k) \\ \mathrm{R}_2(k) + \mathrm{J}_0(k) \\ \mathrm{R}_3(k) + \mathrm{J}_1(k \end{bmatrix};$$

$$3)\quad {}^{4}\overrightarrow{\mathbf{QF}}(k\mid\mathbf{k}) = {}^{4}\vec{\mathbf{R}}(k) - \hat{\mathbf{k}}^{LS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{k}}^{LS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) + \mathrm{J}_3(k) \\ \mathrm{R}_1(k) + \mathrm{J}_2(k) \\ \mathrm{R}_2(k) - \mathrm{J}_1(k) \\ \mathrm{R}_3(k) - \mathrm{J}_0(k) \end{bmatrix},$$

$${}^{4}\overrightarrow{\mathbf{FQ}}(k\mid\mathbf{i}) = {}^{4}\vec{\mathbf{R}}(k) - \hat{\mathbf{k}}^{RS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\mathbf{k}}^{RS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) - \mathrm{J}_3(k) \\ \mathrm{R}_1(k) + \mathrm{J}_2(k) \\ \mathrm{R}_2(k) - \mathrm{J}_1(k) \\ \mathrm{R}_3(k) + \mathrm{J}_0(k \end{bmatrix};$$

$$4)\quad {}^{4}\overrightarrow{\mathbf{QF}}(k\mid\boldsymbol{\mu}) = {}^{4}\vec{\mathbf{R}}(k) + \hat{\boldsymbol{\mu}}^{LS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\boldsymbol{\mu}}^{LS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) - \mathrm{J}_3(k) \\ \mathrm{R}_1(k) - \mathrm{J}_2(k) \\ \mathrm{R}_2(k) + \mathrm{J}_1(k) \\ \mathrm{R}_3(k) + \mathrm{J}_0(k) \end{bmatrix},$$

$${}^{4}\overrightarrow{\mathbf{FQ}}(k\mid\boldsymbol{\mu}) = {}^{4}\vec{\mathbf{R}}(k) - \hat{\boldsymbol{\mu}}^{RS}\cdot{}^{4}\vec{\mathbf{J}}(k) =$$

$$= \begin{bmatrix} QF_0(k) \\ QF_1(k) \\ QF_2(k) \\ QF_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) \\ \mathrm{R}_1(k) \\ \mathrm{R}_2(k) \\ \mathrm{R}_3(k) \end{bmatrix} - \hat{\boldsymbol{\mu}}^{RS}\cdot\begin{bmatrix} \mathrm{J}_0(k) \\ \mathrm{J}_1(k) \\ \mathrm{J}_2(k) \\ \mathrm{J}_3(k) \end{bmatrix} = \begin{bmatrix} \mathrm{R}_0(k) - 0{,}58\left(\mathrm{J}_1(k) + \mathrm{J}_2(k) + \mathrm{J}_3(k)\right) \\ \mathrm{R}_1(k) - 0{,}58\left(-\mathrm{J}_0(k) - \mathrm{J}_2(k) + \mathrm{J}_3(k)\right) \\ \mathrm{R}_2(k) - 0{,}58\left(-\mathrm{J}_0(k) + \mathrm{J}_1(k) - \mathrm{J}_3(k)\right) \\ \mathrm{R}_3(k) - 0{,}58\left(-\mathrm{J}_0(k) - \mathrm{J}_1(k) + \mathrm{J}_2(k)\right) \end{bmatrix},$$

where $\sqrt{3}/3 \approx 0{,}58$.

From the expression (5) and (6) follows that quaternion Fourier transforms are represented by ordinary Fourier transform as

$$Q\mathrm{F}^{(\boldsymbol{\gamma},\boldsymbol{\theta})} = \frac{1}{2}\Big[\left(\mathrm{F} + \mathrm{F}^*\right) - i\circ\mathbf{Diag}\left\{\hat{\boldsymbol{\mu}}^{LS}(\gamma_0,\theta_0),\hat{\boldsymbol{\mu}}^{LS}(\gamma_1,\theta_1),\ldots,\hat{\boldsymbol{\mu}}^{LS}(\gamma_{N-1},\theta_{N-1})\right\}\circ\left(\mathrm{F} - \mathrm{F}^*\right)\Big],$$

$$\mathrm{F}Q^{(\boldsymbol{\gamma},\boldsymbol{\theta})} = \frac{1}{2}\Big[\left(\mathrm{F} + \mathrm{F}^*\right) - i\circ\left(\mathrm{F} - \mathrm{F}^*\right)\circ\mathbf{Diag}\left\{\hat{\boldsymbol{\mu}}^{RS}(\gamma_0,\theta_0),\hat{\boldsymbol{\mu}}^{RS}(\gamma_1,\theta_1),\ldots,\hat{\boldsymbol{\mu}}^{RS}(\gamma_{N-1},\theta_{N-1})\right\}\Big].$$

This fact makes their interpretation, their analysis, and their implementation almost trivial.

5. Conclusion

In this paper, we have shown a new unified approach to the many-parametric representation of complex and quaternion Fourier transforms. Defined representation of many-parameter quaternion Fourier transforms (MPQFTs) depend on finite set of free parameters, which could be changed independently of one another. For each fixed values of parameter we get the unique orthogonal transform. We develop novel Intelligent OFDM-telecommunication systems based on multi-parameter quaternion Fourier transforms. The new systems use inverse MPQFT (IMPQFT) for modulation at the transmitter and direct MPFQT (DMPFQT) for demodulation at the receiver.

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