Matrix WKB solution for electromagnetic waves in an inhomogeneous gyrotropic medium

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Abstract. A matrix solution 4x4 was obtained within the framework of the Wentzel–Kramers–Brillouin method for inclined incidence on a plane electromagnetic wave on an inhomogeneous gyrotropic layer in an external magnetic field with a direction changing in space. Using the obtained matrix solution, the absolute values of the reflectance matrix coefficients and the energy reflection coefficients for the s and p polarized waves are calculated. The dependence of the off-diagonal coefficients of the reflection matrix on the angle of the total rotation of the external magnetic field strength vector within the layer is shown.

Keywords: gyrotropy, inhomogeneous medium, 4x4 method, WKB method, light polarization, Reflection Matrix.

1. Introduction

Planar structures are used in the manufacture of various optical devices: polarizers, light modulators, high and low reflection coatings, optical filters, phase compensators, signal delay lines, optical splitters, VICSELs. The use of unusual optical properties of artificial or natural materials allows not only to direct light, but also to control its characteristics: direction of propagation, localization, polarization and energy. In this paper, we propose a method for calculating the vectors of electromagnetic-wave fields in a plane layer of a gyrotropic plasma. The plasma is in an external magnetic field, the direction of which varies as you move away from the boundaries of the layer.

2. The solution matrix for the inhomogeneous gyrotropic layer

Let us consider an oblique incidence of a plane electromagnetic wave (EMW) on a layer of an ionospheric plasma, which we will approximately assume to be flat. The plasma is in an external magnetic field $\vec{H}_{ext}(z)$ and is gyrotropic; the angle τ between the plane of incidence and the direction of the vector $\vec{H}_{ext}$ is arbitrary and depends on the coordinate z, as shown in Figure 1. The permittivity tensor of an inhomogeneous plasma in an external magnetic field also depends on the z coordinate and has the form [1]:

$$\hat{\varepsilon}=\begin{pmatrix} 1-\frac{\omega_p^2}{\omega^2-\omega_H^2}\sin^2\tau-\frac{\omega_p^2}{\omega^2}\cos^2\tau & \frac{-\omega_p^2\omega_H^2}{2\omega^2\left(\omega^2-\omega_H^2\right)}\sin 2\tau & i\frac{\omega_p^2\omega_H}{\left(\omega^2-\omega_H^2\right)\omega}\sin\tau \\ \frac{-\omega_p^2\omega_H^2}{2\omega^2\left(\omega^2-\omega_H^2\right)}\sin 2\tau & 1-\frac{\omega_p^2}{\omega^2-\omega_H^2}\cos^2\tau-\frac{\omega_p^2}{\omega^2}\sin^2\tau & i\frac{\omega_p^2\omega_H}{\left(\omega^2-\omega_H^2\right)\omega}\cos\tau \\ -i\frac{\omega_p^2\omega_H}{\left(\omega^2-\omega_H^2\right)\omega}\sin\tau & -i\frac{\omega_p^2\omega_H}{\left(\omega^2-\omega_H^2\right)\omega}\cos\tau & 1-\frac{\omega_p^2}{\omega^2-\omega_H^2} \end{pmatrix}. \quad (1)$$

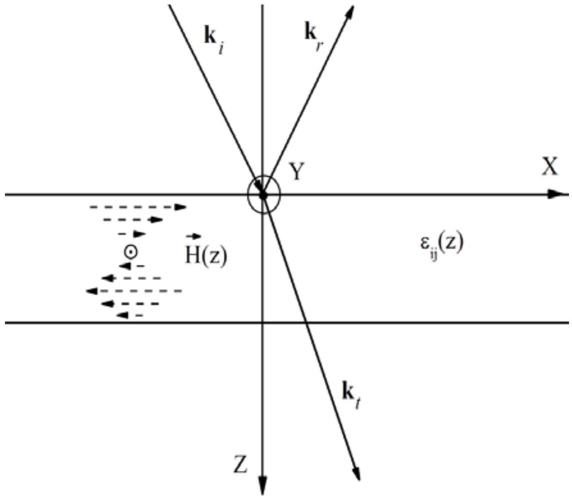


Figure 1. The plane of incidence of an EMW on an inhomogeneous gyrotropic plasma layer.

Here $\omega_p=\sqrt{\frac{4\pi e^2 N_e}{m}}$ – is the plasma frequency, $\omega_H=\frac{eH_{ext}}{mc}$ – is the gyroscopic frequency. The components of the permittivity tensor $\hat{\varepsilon}$ are variable, since the angle τ is variable. The change in the projections of the vectors $\vec{E}$ and $\vec{H}$ in a medium having optical properties (1) will be determined by a system of four ordinary differential equations (ODE), which follows from the Maxwell equations:

$$\frac{d}{dz}\begin{pmatrix} E_y \\ H_x \\ H_y \\ E_x \end{pmatrix}=ik_0\begin{pmatrix} 0 & -\mu & 0 & 0 \\ -\left(\varepsilon_{22}-\frac{\varepsilon_{23}\varepsilon_{32}}{\varepsilon_{33}}-\frac{\alpha^2}{\mu}\right) & 0 & \frac{\alpha\varepsilon_{23}}{\varepsilon_{33}} & -\left(\varepsilon_{21}-\frac{\varepsilon_{23}\varepsilon_{31}}{\varepsilon_{33}}\right) \\ \left(\varepsilon_{12}-\frac{\varepsilon_{13}\varepsilon_{32}}{\varepsilon_{33}}\right) & 0 & -\frac{\alpha\varepsilon_{13}}{\varepsilon_{33}} & \left(\varepsilon_{11}-\frac{\varepsilon_{13}\varepsilon_{31}}{\varepsilon_{33}}\right) \\ -\frac{\alpha\varepsilon_{32}}{\varepsilon_{33}} & 0 & \mu-\frac{\alpha^2}{\varepsilon_{33}} & -\frac{\alpha\varepsilon_{31}}{\varepsilon_{33}} \end{pmatrix}\begin{pmatrix} E_y \\ H_x \\ H_y \\ E_x \end{pmatrix}. \quad (2)$$

Or $\frac{d\vec{Q}}{dz}=ik_0\hat{A}(z)\vec{Q}$. (3)

The value of α is introduced for a brief description of the Snelius law $\alpha={k_{\parallel}}/{k_0}=n(z)\cdot\sin\theta(z)=const$ in the equations; $k_{\parallel}$- is the projection of the wave vector on to the interface z = 0. The components of $\varepsilon_{ij}(z)$ depend on ω, z, on the electron concentration N_e, and on the absolute value of the vector $\vec{H}_{ext}$. We obtain the solution of (2) by the WKB method. We seek the proper solutions of the system in the form: $\exp\left(ik_0\sigma(z)\right)$, where $\sigma(z)=\sigma_0(z)+\frac{\sigma_1(z)}{ik_0}+\ldots$. The analysis of tensor (1) and system (2) shows that $\varepsilon_{12}\varepsilon_{33}-\varepsilon_{13}\varepsilon_{32}=\varepsilon_{21}\varepsilon_{33}-\varepsilon_{23}\varepsilon_{31}$, $\varepsilon_{23}=-\varepsilon_{32}$, $\varepsilon_{13}=-\varepsilon_{31}$. Therefore, the matrix $\hat{A}(z)$ is transformed to the form:

$$\hat{A}(z)=\begin{pmatrix} 0 & -b & 0 & 0 \\ -c & 0 & p & -h \\ h & 0 & -d & e \\ p & 0 & f & d \end{pmatrix}. \tag{4}$$

The parameters *b, c, d, e, f, h, p* in (4) are functions of the *z* coordinate. The matrix $\hat{A}(z)$ for arbitrary z has four eigenvalues λ_i:

$$\lambda_{1,2}=\pm\sqrt{\frac{d^2+ef+bc+\sqrt{\left(d^2+ef-bc\right)^2+4bh\left(fh-dp\right)-4bp\left(dh+ep\right)}}{2}}, \tag{5}$$

$$\lambda_{3,4}=\pm\sqrt{\frac{d^2+ef+bc-\sqrt{\left(d^2+ef-bc\right)^2+4bh\left(fh-dp\right)-4bp\left(dh+ep\right)}}{2}}. \tag{6}$$

These values correspond to four waves for which the vectors $\vec{Q}$ have the form:

$$\left(E_{y,i} \quad H_{x,i} \quad H_{y,i} \quad E_{x,i}\right)^T e^{ik_0\int_0^z \lambda_i(\xi)d\xi}, \tag{7}$$

Waves propagating in the forward direction correspond to the eigenvalues of $\lambda_{1,3}$, and to waves propagating in the opposite direction of the value - $\lambda_{2,4}$. The waves corresponding to $\lambda_{1,2}$ are extraordinary, and the waves corresponding to $\lambda_{3,4}$ - are ordinary. In the particular case when $\tau=90^0$ and the vector $\vec{H}_{ext}$ are perpendicular to the plane of incidence of the EMW, then: $\lambda_{1,2}=\pm\sqrt{d^2+ef}$, $\lambda_{3,4}=\pm\sqrt{bc}$, and the solutions of the ODE system (2) are waves of s- and p-polarization. The matrix solution for these two waves in an inhomogeneous anisotropic plasma was obtained in [2] by the Wentzel–Kramers–Brillouin (WKB) method. Of the four functions, the form (7), and the equations of system (1) for the general case, there follows the fundamental matrix of the solution (FMS) $\hat{Y}(z)$, of the form [3]:

$$\hat{Y}(z)=\hat{F}\cdot diag\left[e^{ik_0\int_0^z \lambda_1(\xi)d\xi},\ldots,\ e^{ik_0\int_0^z \lambda_4(\xi)d\xi}\right], \tag{8}$$

and the Cauchy matrix: $\hat{N}(z,z_0)=\hat{Y}(z)\hat{Y}^{-1}(z_0)$ [5]. Formula (8) gives the initial approximation in the WKB method, each eigenvalue (5)-(6) corresponds to the function $\sigma_{0,k}=\int_0^z \lambda_k(\xi)d\xi$. To find the FMS, we first write down the solutions for the projections H_x and E_x with considering (8):

$$H_x=S_1e^{ik_0\int_0^z \lambda_1(\xi)d\xi}+S_2e^{ik_0\int_0^z \lambda_2(\xi)d\xi}, \tag{9}$$

$$E_x=S_3e^{ik_0\int_0^z \lambda_3(\xi)d\xi}+S_4e^{ik_0\int_0^z \lambda_4(\xi)d\xi}. \tag{10}$$

For the projections E_y and H_y from the system (2) we obtain system of linear equations:

$$\begin{cases} ik_0 cE_x - ik_0 pH_y = -\dfrac{d}{dz}H_x + ik_0 hE_x, \\ ik_0 pE_x + ik_0 fH_y = \dfrac{d}{dz}E_x - ik_0 dE_x. \end{cases} \quad (11)$$

Then $\hat{Y}(z)$ takes the form:

$$\hat{Y}(z) = \begin{pmatrix} -\dfrac{S_1 f\lambda_1}{\Delta(z)} e^{ik_0\int_0^z \lambda_1(\xi)d\xi} & \dfrac{S_2 f\lambda_1}{\Delta(z)} e^{-ik_0\int_0^z \lambda_1(\xi)d\xi} & S_3 \dfrac{hf + p(\lambda_3 - d)}{\Delta(z)} e^{ik_0\int_0^z \lambda_3(\xi)d\xi} & S_4 \dfrac{hf - p(\lambda_3 + d)}{\Delta(z)} e^{-ik_0\int_0^z \lambda_3(\xi)d\xi} \\ S_1 e^{ik_0\int_0^z \lambda_1(\xi)d\xi} & S_2 e^{-ik_0\int_0^z \lambda_1(\xi)d\xi} & 0 & 0 \\ \dfrac{S_1 p\lambda_1}{\Delta(z)} e^{ik_0\int_0^z \lambda_1(\xi)d\xi} & \dfrac{S_2 p\lambda_1}{\Delta(z)} e^{-ik_0\int_0^z \lambda_1(\xi)d\xi} & S_3 \dfrac{c(\lambda_3 - d) - ph}{\Delta(z)} e^{ik_0\int_0^z \lambda_3(\xi)d\xi} & -S_4 \dfrac{c(\lambda_3 + d) + ph}{\Delta(z)} e^{-ik_0\int_0^z \lambda_3(\xi)d\xi} \\ 0 & 0 & S_3 e^{ik_0\int_0^z \lambda_3(\xi)d\xi} & S_4 e^{-ik_0\int_0^z \lambda_3(\xi)d\xi} \end{pmatrix}. \quad (12)$$

Here $\Delta(z) = c(z)f(z) + p^2(z)$.

$$\hat{Y}^{-1}(0) = \begin{pmatrix} -\dfrac{c(0)}{2S_1\lambda_1(0)} & \dfrac{1}{2S_1} & \dfrac{p(0)}{2S_1\lambda_1(0)} & \dfrac{h(0)}{2S_1\lambda_1(0)} \\ \dfrac{c(0)}{2S_2\lambda_1(0)} & \dfrac{1}{2S_2} & -\dfrac{p(0)}{2S_2\lambda_1(0)} & -\dfrac{h(0)}{2S_2\lambda_1(0)} \\ \dfrac{p(0)}{2S_3\lambda_3(0)} & 0 & \dfrac{f(0)}{2S_3\lambda_3(0)} & \dfrac{d(0) + \lambda_3(0)}{2S_3\lambda_3(0)} \\ -\dfrac{p(0)}{2S_4\lambda_3(0)} & 0 & -\dfrac{f(0)}{2S_4\lambda_3(0)} & \dfrac{-d(0) + \lambda_3(0)}{2S_4\lambda_3(0)} \end{pmatrix}. \quad (13)$$

In the formulas (12) - (13), the values of the variable coefficients of the matrix of the system (2) in the plane z = const are assumed to be the values *c, d, f, h, p,* λ_i, Δ(z) and the values *c(0), d(0), f(0), h(0), p(0),* $\lambda_i(0)$ – are the values of the same quantities in the z = 0 plane. The coefficients n_{ij} of the Cauchy matrix are calculated by the formula: $\hat{N}(z,0) = \hat{Y}(z)\hat{Y}^{-1}(0)$. Matrix methods allow the boundary conditions for the fields at the media interfaces to be stitched, the fields in thin films and waveguides can be calculated [4]. The Cauchy matrix $\hat{N}(z,0)$ makes it possible to stitch solutions on the boundary of the layer and calculate the reflection matrix from the values of n_{ij} [5]:

$$\hat{R} = \begin{pmatrix} R_{ss} & R_{sp} \\ R_{ps} & R_{pp} \end{pmatrix}. \quad (14)$$

When the inhomogeneously gyrotropic layer is reflected at the boundary, the polarization and amplitude of the wave vary according to the rule:

$$\begin{pmatrix} E_{s,r} \\ E_{p,r} \end{pmatrix} = \begin{pmatrix} R_{ss} & R_{sp} \\ R_{ps} & R_{pp} \end{pmatrix} \begin{pmatrix} E_{s,i} \\ E_{p,i} \end{pmatrix}. \quad (15)$$

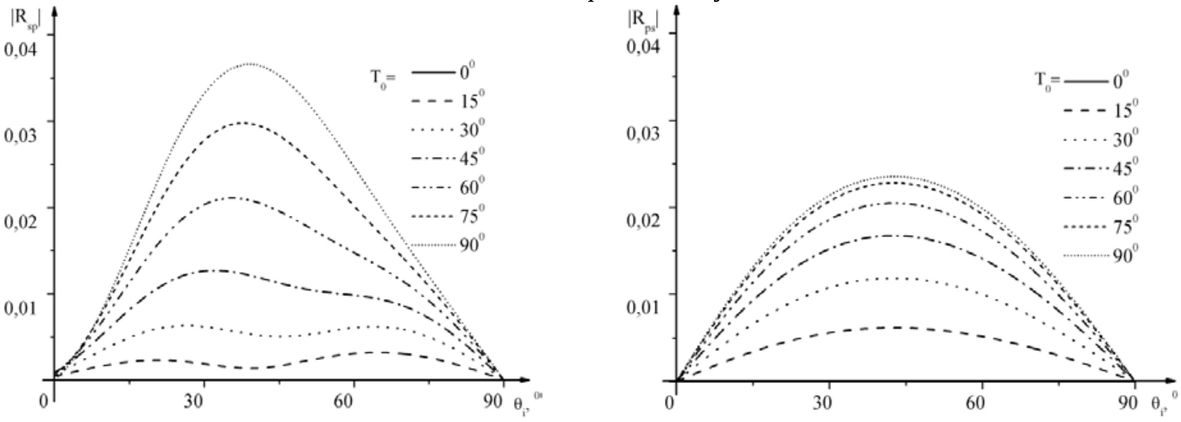
3. Calculations

For a plane layer of thickness $d = 5\lambda_0$, with parameters: $\mu = 1$, whose dielectric permittivity tensor has the form (1), the modues of the coefficients of the reflection matrix (14) were calculated. The ratio of the plasma frequency to the radiation frequency: $u = \omega_p / \omega = 2.5$, the ratio $W = \omega_H / \omega = 0.03$, the angle τ depends on the coordinate z: $\tau = T_0 \cdot z/d$, the angle T_0 takes the values: 0^0, 15^0, $,90^0$. The absolute values of $|R_{ss}|$ and $|R_{pp}|$ are close to unity. The dependencies of $|R_{sp}|$ and $|R_{ps}|$ are shown in the Figure 2. With increasing rotation angle T_0 in the medium, the values of $|R_{sp}|$ and $|R_{ps}|$ increase. The energy reflection coefficients for s- and p-polarization waves were calculated from the formulas:

$$\Re_S = \left(R_{SS} + R_{SP}\right)\left(R_{SS} + R_{SP}\right)^*, \tag{15}$$

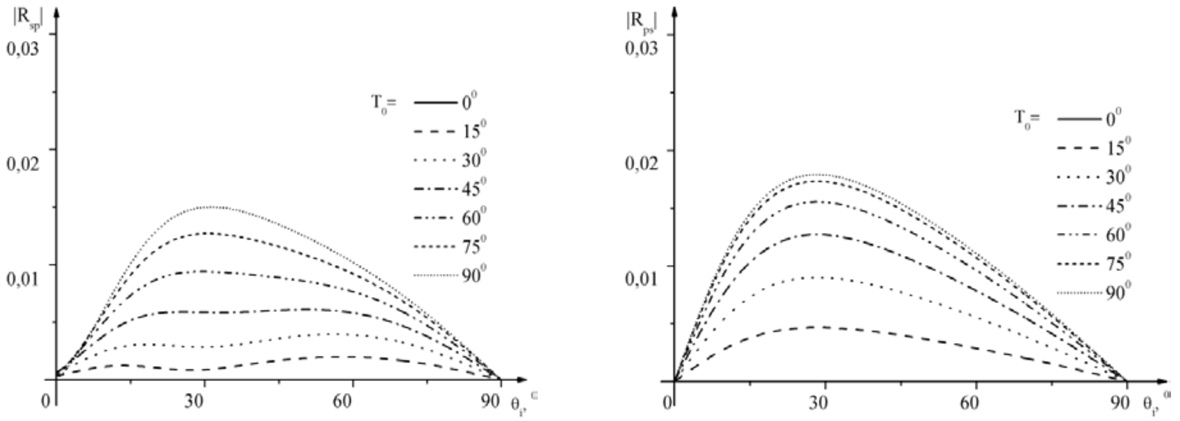
$$\Re_P = \left(R_{PS} + R_{PP}\right)\left(R_{PS} + R_{PP}\right)^* \tag{16}$$

The calculation showed that their values are equal to unity.



a) b)

Figure 2. Absolute values of the reflection coefficients a) R_{sp}, b) R_{ps}.



a) b)

Figure 3. Absolute values of the reflection coefficients a) R_{sp}, b) R_{ps}.

For a layer, thickness $d = 5\lambda_0$ with $u = \omega_p / \omega = 1.5$ and $W = \omega_H / \omega = 0.015$ the modules of the coefficients of the matrix $\hat{R}$ were calculated for a linear change in the angle τ. The angle T_0 assumed the values: 0^0, 15^0, , 90^0. The dependencies of $|R_{sp}|$ and $|R_{ps}|$ are shown in the Figure 3. With increasing T_0 the angular dependences of $|R_{sp}|$ and $|R_{ps}|$ increase. The calculation showed that for the given parameters, complete internal reflection is performed. The calculation showed that for the given parameters, total internal reflection is performed. This is in good agreement with the known results: in both cases the frequency ω was less than the plasma frequency ω_p. The values of the energy reflection coefficients $\Re_S$ and $\Re_P$ are equal to one.

4. Conclusions

A matrix method for calculating fields in an inhomogeneous gyrotropic layer is obtained. The coefficients of the reflection matrix for the gyrotropic plasma layer are calculated, the optical properties of which depend on the transverse coordinate inside the layer. It is shown that when light is reflected from a gyrotropic medium with torsion, cross-polarized components appear.

5. References

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