Many-lateral MIMO-filters for hyperspectral image filtering

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Abstract. In the paper, we investigate effectiveness of modified many-factor (bilateral, tri-, and four-lateral) denoising MIMO-filters for grey, color, and hyperspectral image procession. Conventional bilateral filter performs merely weighted averaging of the local neighborhood pixels. The weight includes two components: spatial and radiometric ones. The first component measures the geometric distances between the center pixel and local neighborhood ones. The second component measures the radiometric distance between the values of the center pixel and local neighborhood ones. Noise affects all pixels even onto the centre one used as a reference for the tonal filtering. Thus, the noise affecting the centre pixel has a disproportionate effect onto the result. This suggests the first modification: the center pixel is replaced by the weighted average (with some estimate of the true value) of the neighborhood pixels contained in a window around it. The second modification uses the matrix-valued weights. They include four components: spatial, radiometric, inter-channel weights, and radiometric inter-channel ones. The fourth weight measures the radiometric distance (for grey-level images) between the inter-channel values of the center scalar-valued channel pixel and local neighborhood channel ones.

1. Introduction

We develop a conceptual framework and design methodologies for multichannel image many-lateral (bilateral, 3-, and 4-lateral) aggregation filters with assessment capability. The term “multichannel” (multicomponent, multispectral, hyperspectral) image is used for an image with more than one component. They are composed of a series of images in different optical bands at wavelengths $\lambda_1, \lambda_2, ..., \lambda_K$, called spectral channels: $\mathbf{f}(\mathbf{x}) = \left(f_{\lambda_1}(\mathbf{x}), f_{\lambda_2}(\mathbf{x}), ..., f_{\lambda_K}(\mathbf{x}, y) \right) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_K(\mathbf{x}) \right)$, where K is the number of different optical channels, *i.e.*, $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^K$, where $\mathbf{R}^K$ is multicolor space. The bold font for $\mathbf{f}(\mathbf{x})$ emphasizes the fact that images may be multichannel.Each pixel in $\mathbf{f}(\mathbf{x})$, therefore, represents the spectrum at the wavelengths $\lambda_1, \lambda_2, ..., \lambda_K$ of the observed scene at point $\mathbf{x} = (i, j) \in \mathbf{Z}^2$.

Let us introduce the observation model and notions used throughout the paper. We consider noised image in the form $\mathbf{f}(\mathbf{x}) = \mathbf{s}(\mathbf{x}) + \mathbf{n}(\mathbf{x})$, where $\mathbf{s}(\mathbf{x})$ is the original grey-level image and $\mathbf{n}(\mathbf{x})$ denotes the noise introduced into $\mathbf{s}(\mathbf{x})$ to produce the corrupted image $\mathbf{f}(\mathbf{x})$ and where $\mathbf{x} = (i, j) \in \mathbf{Z}^2$ (or $\mathbf{x} = (i, j, k) \in \mathbf{Z}^3$) is a 2D (or 3D) coordinates that belong to the image domain and represent the pixel location. If $\mathbf{x} \in \mathbf{Z}^2, \mathbf{Z}^3$ then $\mathbf{f}(\mathbf{x}), \mathbf{s}(\mathbf{x}), \boldsymbol{\eta}(\mathbf{x})$ are 2D or 3D images, respectively. The aim of image enhancement is to reduce the noise as much as possible or to find a method which, given $\mathbf{s}(\mathbf{x})$, derives an image $\hat{s}(\mathbf{x})$ as close as possible to the original $\mathbf{s}(\mathbf{x})$, subjected to a suitable optimality criterion.

The standard bilateral filter (BF) [1-9] with a square N-cellular window $M(\mathbf{x})$ is located at $\mathbf{x}$, the weighted average of pixels in the moving window replaces the central pixel

$$\hat{\mathbf{s}}(\mathbf{x}) = \underset{\mathbf{p}\in M(\mathbf{x})}{\mathbf{BilMean}}\left[w(\mathbf{x},\mathbf{p})\cdot\mathbf{f}(\mathbf{p})\right] = \frac{1}{k(\mathbf{x})}\sum_{\mathbf{p}\in M(\mathbf{x})} w(\mathbf{x},\mathbf{p})\cdot\mathbf{f}(\mathbf{p}), \qquad (1)$$

where $\hat{s}(\mathbf{x})$ is the filtered image and $k(\mathbf{p})$ is the normalization factor

$$k(\mathbf{x}) = \sum_{\mathbf{p}\in M(\mathbf{x})} w(\mathbf{x},\mathbf{p}). \qquad (2)$$

Equation (1) is simply a normalized weighted average of a neighborhood of a N-cellular window $M(\mathbf{x})$ (*i.e.*, the mask around pixel $\mathbf{x}$, consisting of N pixels).

The scalar-valued weights $w(\mathbf{x},\mathbf{p})$ are computed based on the content of the neighborhood. For pixels $\{\mathbf{f}(\mathbf{p})\}_{\mathbf{p}\in M(\mathbf{x})}$ around the centroid $\mathbf{f}(\mathbf{x})$, the weights $\{w(\mathbf{x},\mathbf{p})\}_{\mathbf{p}\in M(\mathbf{x})}$ are computed by multiplying the following two factors: $w(\mathbf{x},\mathbf{p}) = w_{Sp}(\mathbf{p})\cdot w_{Rn}(\mathbf{x},\mathbf{p}) = w_{Sp}(\|\mathbf{p}\|)\cdot w_{Rn}(\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|_2)$. The weight includes two factors – spatial $w_{Sp}(\|\mathbf{p}\|)$ and radiometric $w_{Rn}(\mathbf{x},\mathbf{p}) = w_{Rn}(\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|_2)$ weights.

The first weight measures the geometric distance $\|\mathbf{p}\|$ between the center pixel $f(\mathbf{x})$ and the pixel $\mathbf{f}(\mathbf{p})$ (note, the centroid $\mathbf{x}$ has the position $\mathbf{0}\in M(\mathbf{x})$ inside of the mask $M(\mathbf{x})$). Here the Euclidean metric $\|\mathbf{p}\|=\|\mathbf{p}\|_2$ is applied. This way, close-by pixels influence the final result more than distant ones. The second weight measures the radiometric distance between the values of the center sample $f(\mathbf{x})$ and the pixel $\mathbf{f}(\mathbf{p})$, and again, the Euclidean metric $\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|_2$ is chosen, too. Therefore, pixels with close-by values tend to influence the final result more than those having distant value. The traditional bilateral filter uses the Gaussian or Laplacian kernels for both spatial and range (or tonal) filtering. The classical bilateral filter is a non-linear filter which takes into account local image information in order to build a kernel which smoothes without smoothing across edges.

This paper considers two natural extensions to the bilateral filter. Firstly, instead of the center pixel $\mathbf{f}(\mathbf{x})$ in $w_{Rn}(\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|)$, we use a certain mean or median $\bar{\mathbf{f}}(\mathbf{x})$ (for example, the *suboptimal Fréchet median* $\bar{\mathbf{f}}(\mathbf{x}) = \bar{\mathbf{f}}_{sopt}(\mathbf{x})$ [11]) of a neighborhood of a N-cellular window $M(\mathbf{x})$ for calculating of weighs $w_{Rn}(\mathbf{x},\mathbf{p}) = w_{Rn}(\|\bar{\mathbf{f}}_{sopt}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|_2)$. Secondly, instead of a scale-valued weigh, we use a matrix-valued one

$$\hat{\mathbf{s}}(\mathbf{x}) = \underset{\mathbf{p}\in M(\mathbf{x})}{\mathbf{BilMean}}\left[\mathbf{W}(\mathbf{x},\mathbf{p})\cdot\mathbf{f}(\mathbf{p})\right], \qquad (3)$$

where $\mathbf{W}(\mathbf{x},\mathbf{p})$ are the matrix-valued weighs.

2. The first modification of bilateral filters

In this modification we use the *suboptimal Fréchet median* $\bar{\mathbf{f}}_{sopt}(\mathbf{x})$ for calculating of weighs $w_{Rn}(\mathbf{x},\mathbf{p})$ instead of the center pixel $\mathbf{f}(\mathbf{x})$ in w_{Rn}: $w_{Rn}(\|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|)\to w_{Rn}(\|\bar{\mathbf{f}}_{sopt}(\mathbf{x})-\mathbf{f}(\mathbf{p})\|)$. Let $\langle\mathbf{R}^K,\rho\rangle$ be a metric spaces, where ρ is a distance function (*i.e*, $\rho:\mathbf{R}^K\times\mathbf{R}^K\to\mathbf{R}^+$). Let $w_1,w_2,\ldots,w_N$ be N weights summing to 1 and let $\{\mathbf{f}^1,\mathbf{f}^2,\ldots,\mathbf{f}^N\}=\mathbf{D}\subset\mathbf{R}^K$ be N pixels in the N-cellular window $M(\mathbf{x})$.

Definition 1 [10, 11]. The *optimal Fréchet point* associated with the metric ρ, is the point $\mathbf{f}_{opt}\in\mathbf{R}^K$, that minimizes the Fréchet cost function (FCF) $\sum_{i=1}^{N} w_i\rho(\mathbf{f},\mathbf{f}^i)$ (the weighted sum distances from an arbitrary point $\mathbf{f}$ to each point $\mathbf{f}^1,\mathbf{f}^2,\ldots,\mathbf{f}^N\in\mathbf{R}^K$). It is formally defined as:

$$\bar{\mathbf{f}}_{opt} = \mathbf{FrechMed}\left(\rho\,|\,\mathbf{f}^1,\mathbf{f}^2,\ldots,\mathbf{f}^N\right) = \underset{\mathbf{f}\in\mathbf{R}^K}{\mathbf{arg\,min}}\sum_{i=1}^{N} w_i\rho(\mathbf{f},\mathbf{f}^i). \qquad (4)$$

Note that **argmin** means the argument, for which the sum is minimized. So, the vector-valuedmedian of a discrete set of sample points in a Euclidean space $\mathbf{R}^K$ is the point minimizing the sum of distances to the sample points. This generalizes the ordinary median, which has the property of minimizing the sum of distances for one-dimensional data, and provides a central tendency higher dimensions.

In computation point of view, it is better to restrict the search domain from $\mathbf{R}^K$ until the set $\mathbf{D} = \{\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\} \subset \mathbf{R}^K$. In this case, we obtain definition of the *suboptimal Fréchet point* or the *optimal vector Fréchet median*.

Definition 2 [10,11]. The *suboptimal weighted Fréchet point* or *optimal Fréchet median* associated with the metric ρ is the point, $\overline{\mathbf{f}}_{sopt} \in \{\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\} \subset \mathbf{R}^K$, that minimizes the FCF over the the restrict search domain $\mathbf{D} \subset \mathbf{R}^K$:

$$\overline{\mathbf{f}}_{sopt} = \mathbf{FrechMed}\left(\rho \mid \mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\right) = \underset{\mathbf{f} \in \mathbf{D}}{\mathbf{argmin}} \sum_{i=1}^{N^2} w_i \rho\left(\mathbf{f}, \mathbf{f}^i\right). \quad (2)$$

Example 1. If observation data are real numbers, *i.e.*, $f^1, f^2, ..., f^N \in \mathbf{R}$ and distance function is the city distance $\rho(f,g) = \rho_1(f,g) = \left|f - g\right|$, then the optimal Fréchet point (4) and optimal Fréchet medians (5) for grey-level pixels $f^1, f^2, ..., f^N \in \mathbf{R}$ to be the classical *Fréchet point* and *median*, respectively, *i.e.*,

$$\overline{\mathbf{f}}_{opt} = \mathbf{FrechPt}\left(\rho_1 \mid f^1, f^2, ..., f^{N^2}\right) = \underset{f \in \mathbf{R}}{\mathbf{arg\,min}} \sum_{i=1}^{N} \left|f - f^i\right|, \quad (3)$$

$$\overline{\mathbf{f}}_{sopt} = \mathbf{FrechMed}\left(\rho_1 \mid f^1, f^2, ..., f^N\right) = \underset{\mathbf{f} \in \mathbf{D}}{\mathbf{argmin}} \sum_{i=1}^{N} \left|f - f^i\right| = \mathbf{Med}\left(f^1, f^2, ..., f^N\right). \quad (4)$$

Example 2. If observation data are vectors, *i.e.*, $\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N \in \mathbf{R}^K$, and distance function is the city distance $\rho(\mathbf{f}, \mathbf{g}) = \rho_1(\mathbf{f}, \mathbf{g})$, then the optimal Fréchet point (4) and optimal Fréchet medians (5) for vectors $\mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N \in \mathbf{R}^K$ to be the *Fréchet point* and *theFréchet vector median*, associated with the same metric $\rho_1(\mathbf{f}, \mathbf{g})$,

$$\mathbf{f}_{opt} = \mathbf{FrechPt}\left(\rho_1 \mid \mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\right) = \underset{\mathbf{f} \in \mathbf{R}^K}{\mathbf{argmin}} \left(\sum_{i=1}^{N} \|\mathbf{f} - \mathbf{f}^i\|_1\right), \quad (5)$$

$$\overline{\mathbf{f}}_{sopt} = \mathbf{FrechMed}\left(\rho_1 \mid \mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\right) = \underset{f \in \mathbf{D}}{\mathbf{argmin}} \left(\sum_{i=1}^{N} \|\mathbf{f} - \mathbf{f}^i\|_1\right) = \mathbf{VecMed}\left(\rho_1 \mid \mathbf{f}^1, \mathbf{f}^2, ..., \mathbf{f}^N\right). \quad (6)$$

Now we use Fréchet median $\overline{\mathbf{f}}_{sopt}$ for calculating radiometric weights $w_{Rn}(\mathbf{x}, \mathbf{p}) = w_{Rn}\left(\|\overline{\mathbf{f}}_{sopt}(\mathbf{x}) - \mathbf{f}(\mathbf{p})\|_2\right)$. The modified bilateral filter (MBF) is given as

$$\hat{\mathbf{s}}(\mathbf{x}) = \underset{\mathbf{p} \in M(\mathbf{x})}{\mathbf{BilMean}}\left[w(\mathbf{x}, \mathbf{p}) \cdot \mathbf{f}(\mathbf{p})\right] = \frac{1}{k(\mathbf{x})} \sum_{\mathbf{p} \in M(\mathbf{x})} w_{Sp}(\|\mathbf{p}\|) \cdot w_{Rn}\left(\|\overline{\mathbf{f}}_{sopt}(\mathbf{x}) - \mathbf{f}(\mathbf{p})\|_2\right) \cdot \mathbf{f}(\mathbf{x}), \quad (10)$$

where $\hat{\mathbf{s}}(\mathbf{x})$ is the filtered image.

3. The second modification. Four-lateral MIMO-filter

In the case of the multichannel images, processed data are vector-valued $\mathbf{f}(\mathbf{x}) : \mathbf{R}^2 \to \mathbf{R}^K$: $\mathbf{f}(\mathbf{x}) = \left(f_1(\mathbf{x}), f_2(\mathbf{x}), ..., f_K(\mathbf{x})\right) = [f_c(\mathbf{x})]_{c=1}^K$. By this reason, we must use matrix-valuedweights $\{\mathbf{W}(\mathbf{x}, \mathbf{p})\}_{\mathbf{p} \in M(\mathbf{x})}$, where $\mathbf{W}(\mathbf{x}, \mathbf{p})$ is a $(K \times K)$-matrix, and K is the number of different channels in $\mathbf{f}(\mathbf{x}) : \mathbf{R}^2 \to \mathbf{R}^K$. The 4-factor MIMO-filter suggests a weighted average of pixels in the given image

$$\hat{\mathbf{s}}(\mathbf{x}) = \underset{\mathbf{p} \in M(\mathbf{x})}{\mathbf{MIMO\,^4FactMean}}\left[\overline{\mathbf{W}}(\mathbf{x}, \mathbf{p}) \cdot \mathbf{f}(\mathbf{p})\right] = \frac{1}{\mathbf{diag}\left\{k_1(\mathbf{x}), k_2(\mathbf{x}), ..., k_K(\mathbf{x})\right\}} \sum_{\mathbf{p} \in M(\mathbf{x})} \mathbf{W}(\mathbf{x}, \mathbf{p}) \cdot \mathbf{f}(\mathbf{p}), \quad (7)$$

or in component-wise form

$$\hat{s}_a(\mathbf{x}) = \frac{1}{k_a(\mathbf{x})} \sum_{\mathbf{p}\in M(\mathbf{x})} \sum_{b=1}^{K} w^{ab}(\mathbf{x},\mathbf{p}) \cdot f_b(\mathbf{p}) = \sum_{\mathbf{p}\in M(\mathbf{x})} \sum_{b=1}^{K} \bar{w}^{ab}(\mathbf{x},\mathbf{p}) \cdot f_b(\mathbf{p}), \quad (8)$$

where $\hat{\mathbf{s}}(\mathbf{x})$ is the filtered multichannel image, $\hat{s}_a(\mathbf{x})$ is its ath channel, $\bar{w}^{ab} = w^{ab} / k_a$, $\bar{\mathbf{W}}(\mathbf{x},\mathbf{p}) = \mathbf{diag}\{k_1^{-1}(\mathbf{x}), k_2^{-1}(\mathbf{x}),..., k_K^{-1}(\mathbf{x})\} \cdot \mathbf{W}(\mathbf{x},\mathbf{p})$, $k_a(\mathbf{p})$ is the normalization factor in the ath channel:

$$k_a(\mathbf{x}) = \sum_{\mathbf{p}\in M(\mathbf{x})} \sum_{b=1}^{K} w^{ab}(\mathbf{x},\mathbf{p}) \quad (9)$$

and $\mathbf{diag}\{k_1(\mathbf{x}), k_2(\mathbf{x}),..., k_K(\mathbf{x})\}$ is diagonal matrix with channel normalization factors. Note, that

$$\frac{1}{\mathbf{diag}\{k_1(\mathbf{x}), k_2(\mathbf{x}),..., k_K(\mathbf{x})\}} \mathbf{W}(\mathbf{x},\mathbf{p}) \cdot \mathbf{f}(\mathbf{p}) = \bar{\mathbf{W}}(\mathbf{x},\mathbf{p}) \cdot \mathbf{f}(\mathbf{p}) =$$

$$= \begin{bmatrix} k_1^{-1}(\mathbf{x}) & & \vdots & \\ & k_2^{-1}(\mathbf{x}) & \vdots & \\ \vdots & \vdots & \vdots & \vdots \\ & & \vdots & k_K^{-1}(\mathbf{x}) \end{bmatrix} \begin{bmatrix} w^{11}(\mathbf{x},\mathbf{p}) & w^{12}(\mathbf{x},\mathbf{p}) & \cdots & w^{1K}(\mathbf{x},\mathbf{p}) \\ w^{21}(\mathbf{x},\mathbf{p}) & w^{22}(\mathbf{x},\mathbf{p}) & \cdots & w^{2K}(\mathbf{x},\mathbf{p}) \\ \vdots & \vdots & \vdots & \vdots \\ w^{K1}(\mathbf{x},\mathbf{p}) & w^{K2}(\mathbf{x},\mathbf{p}) & \cdots & w^{KK}(\mathbf{x},\mathbf{p}) \end{bmatrix} \begin{bmatrix} f_1(\mathbf{p}) \\ f_2(\mathbf{p}) \\ \vdots \\ f_K(\mathbf{p}) \end{bmatrix}.$$

The normalized matrix-valued weights $\bar{\mathbf{w}}(\mathbf{x},\mathbf{p})$ are computed based on the content of the neighborhood. For pixels $\mathbf{f}(\mathbf{p}),\ \mathbf{p}\in M(\mathbf{x})$ around the Fréchet centroid $\bar{\mathbf{f}}_{sopt}(\mathbf{x})$, the scalar-valued weights $\bar{w}^{cd}(\mathbf{x},\mathbf{p})$ of the matrices $\bar{\mathbf{W}}(\mathbf{x},\mathbf{p}),\ \mathbf{p}\in M(\mathbf{x})$ are computed by multiplying the following four factors:

$$\bar{w}^{cd}(\mathbf{x},\mathbf{p}) = \bar{w}_{Sp}(\|\mathbf{p}\|) \cdot \bar{w}_{Ch}\left(\left|c-d\right|\right) \cdot \bar{w}_{Rn}\left(\|\bar{\mathbf{f}}_{sopt}(\mathbf{x})) - \mathbf{f}(\mathbf{p})\|_2\right) \cdot \bar{w}_{Rn}\left(\left|\bar{f}_{c,sopt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right).$$

The weight includes four factors: spatial $\bar{w}_{Sp}(\|\mathbf{p}\|)$, inter-channels $\bar{w}_{Ch}\left(\left|c-d\right|\right)$, global radiometric $\bar{w}_{Rn}\left(\|\bar{\mathbf{f}}_{sopt}(\mathbf{x})) - \mathbf{f}(\mathbf{p})\|_2\right)$, and radiometric inter-channels weights $\bar{w}_{Rn}\left(\left|\bar{f}_{c,sopt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)$. The first factor $\bar{w}_{Sp}(\|\mathbf{p}\|)$ measures the geometric distance between the center pixel $\bar{\mathbf{f}}_{sopt}(\mathbf{x})$ and the neighborhood pixels $\mathbf{f}(\mathbf{p}),\ \mathbf{p}\in M(\mathbf{x})$. The second factor $\bar{w}_{Ch}\left(\left|c-d\right|\right)$ measures the spectral (inter-channel) distance. The third factor $w_{Rn}\left(\|\bar{\mathbf{f}}_{sopt}(\mathbf{x}) - \mathbf{f}(\mathbf{p})\|_2\right)$ measures the global radiometric distance between the values of the Fréchet center $\bar{\mathbf{f}}_{sopt}(\mathbf{x})$ and the pixels $\mathbf{f}(\mathbf{p}),\ \mathbf{p}\in M(\mathbf{x})$. The fourth factor $\bar{w}_{Rn}\left(\left|\bar{f}_{c,opt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)$ measures the radiometric distance between the values of the center sample $\bar{f}_{c,sopt}(\mathbf{x})$ of the c-channel and the pixel $f_d(\mathbf{p}),\ \mathbf{p}\in M(\mathbf{x})$ of the d-channel. All weights $\bar{w}_{Rn}^{cd}(\mathbf{x},\mathbf{p}) = \bar{w}_{Rn}\left(\left|\bar{f}_{c,opt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)$ form N radiometric inter-channel $(K \times K)$-matrices

$$\left\{\bar{\mathbf{W}}_{Rn}(\mathbf{x},\mathbf{p})\right\}_{\mathbf{p}\in M(\mathbf{x})} = \left\{\left[\bar{w}_{Rn}^{cd}(\mathbf{x},\mathbf{p})\right]_{c,d=1}^{K}\right\}_{\mathbf{p}\in M(\mathbf{x})} = \left\{\left[\bar{w}_{Rn}\left(\left|\bar{f}_{c,opt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)\right]_{c,d=1}^{K}\right\}_{\mathbf{p}\in M(\mathbf{x})}$$

If N-cellular window is used. We obtain 3-factor MIMO-filters if we are going to use only three ingredients, for example, $\bar{w}^{cd}(\mathbf{x},\mathbf{p}) = \bar{w}_{Sp}(\|\mathbf{p}\|) \cdot \bar{w}_{Rn}\left(\|\bar{\mathbf{f}}_{opt}(\mathbf{x}) - \mathbf{f}(\mathbf{p})\|_2\right) \cdot \bar{w}_{Rn}\left(\left|\bar{f}_{c,opt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)$ or $\bar{w}^{cd}(\mathbf{x},\mathbf{p}) = \bar{w}_{Sp}(\|\mathbf{p}\|) \cdot \bar{w}_{Ch}\left(\left|c-d\right|\right) \cdot \bar{w}_{Rn}\left(\left|\bar{f}_{c,opt}(\mathbf{x}) - f_d(\mathbf{p})\right|\right)$.

4. Simulation results

Some variants of the proposed filters are tested. They are compared on real image "LENA". Noise is added (see Fig. 1) with different the Peak Signal to Noise Ratios (PSNRs). The noised images has 1% noised pixels (PSNR = 25.36 dB), 5% noised pixels (PSNR = 18.34 dB), 10% noised pixels (PSNR = 15.41 dB), 20% noised pixels (PSNR = 12.64 dB), 50% noised pixels (PSNR = 9.22 dB). Fig. 2-4 summarize the results for "Salt and Pepper" noise and bilateral filters with Laplacian weights

$w(f,g)=\exp\left(-\alpha\left|f-g\right|\right)$ for different α = 0.035 (Fig. 2), α = 0.07 (Fig. 3), α = 0.1 (Fig. 4). Fig. 2-4 show the results obtained by the following bilateral filters with $(_3$ -mask

- the classical bilateral filter (1) (**BF3x3**),
- modified bilateral filter (10) (**BF3x3Med**), where $\overline{\mathbf{f}}(\mathbf{x})$ is calculating as classical median in each channel,
- modified bilateral filter (10) (**BF3x3Fr1**), where $\overline{\mathbf{f}}(\mathbf{x})$ are calculating as Fréchet median $\overline{\mathbf{f}}(\mathbf{x})=\overline{\mathbf{f}}_{opt}(\mathbf{x})$ with distance $\rho(\mathbf{f},\mathbf{g})=\rho_1(\mathbf{f},\mathbf{g})$,
- modified bilateral filter (10) (**BF3x3Fr2**), where $\overline{\mathbf{f}}(\mathbf{x})$ are calculating as Fréchet median $\overline{\mathbf{f}}(\mathbf{x})=\overline{\mathbf{f}}_{opt}(\mathbf{x})$ with distance $\rho(\mathbf{f},\mathbf{g})=\rho_2(\mathbf{f},\mathbf{g})$,
- modified bilateral filter (10) (**BF3x3Fr∞**), where $\overline{\mathbf{f}}(\mathbf{x})$ are calculating as Fréchet median $\overline{\mathbf{f}}(\mathbf{x})=\overline{\mathbf{f}}_{opt}(\mathbf{x})$ with distance $\rho(\mathbf{f},\mathbf{g})=\rho_\infty(\mathbf{f},\mathbf{g})$.

It is easy to see that results for all modified bilateral filters are better, compared to the classical bilateral filter **BF3x3.**

a) Original image b) Noised images, PSNR = 25.36 c) Noised images, PSNR = 18.34

d) Noised images, PSNR = 15.41 e) Noised images, PSNR = 12.64 f) Noised images, PSNR = 9.22

Figure 1. Original (a) and noised (b) images; noise: Salt-Pepper; denoised images (c)-(f).

5. Future work

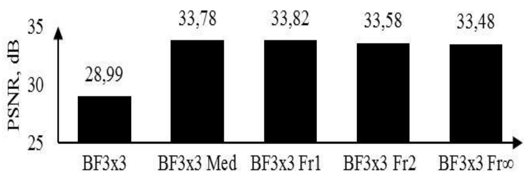
In future, we are going to use in (10) and (11) a generalized average (aggregation) [11-13] instead of ordinary mean. The aggregation problem [11-13] consist in aggregating *n*-tuples of objects all belonging to a given set S, into a single object of the same set S, i.e., $\mathbf{Agg}: S^n \to S$. In the case of mathematical aggregation operator the set S is an interval of the real $S=[0,1]\subset \mathbf{R}$ or integer numbers $S=[0,255]\subset \mathbf{Z}$. In this setting, an AO is simply a function, which assigns a number y to any N-tuple $(x_1,x_2,...,x_N)$ of numbers: $y=\mathbf{Aggreg}(x_1,x_2,...,x_N)$ that satisfies:

1) $\mathbf{Agg}(x)=x$,

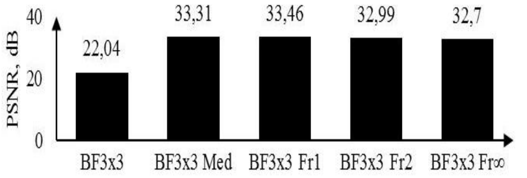
2) $\mathbf{Agg}(\underbrace{a,a,...,a}_{N})=a$. In particular, $\mathbf{Agg}(0,0,...,0)=0$ and $\mathbf{Agg}(1,1,...,1)=1$, or

$\mathbf{Agg}(255,255,...,255)=255$.

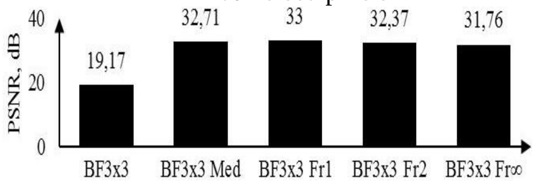
3) $\mathbf{min}(x_1,x_2,...,x_N)\le \mathbf{Agg}(x_1,x_2,...,x_N)\le \mathbf{max}(x_1,x_2,...,x_N)$.



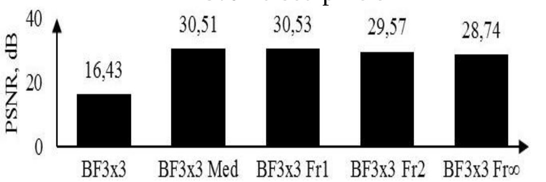
a) Filtration of “Salt and Pepper” noise with 1% noised pixels



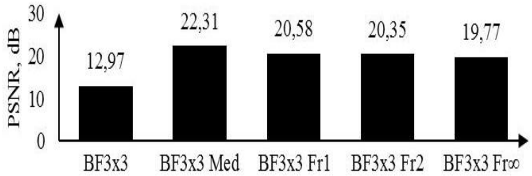
b) Filtration of “Salt and Pepper” noise with 5% noised pixels



c) Filtration of “Salt and Pepper” noise with 10% noised pixels

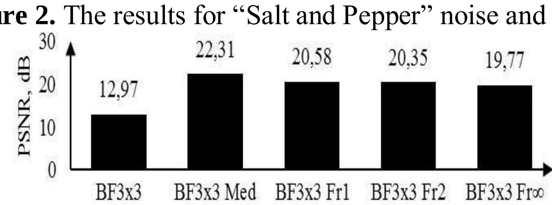


d) Filtration of “Salt and Pepper” noise with 20% noised pixels

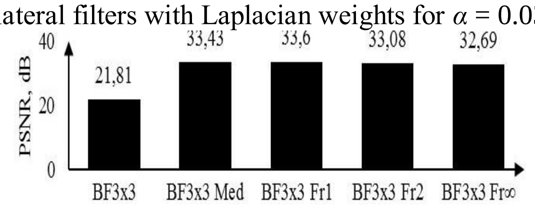


e) Filtration of “Salt and Pepper” noise with 50% noised pixels

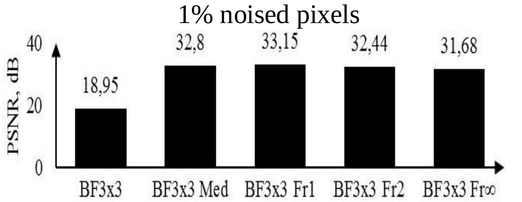
Figure 2. The results for “Salt and Pepper” noise and bilateral filters with Laplacian weights for $\alpha = 0.035$.



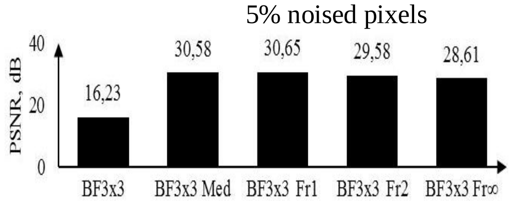
a) Filtration of “Salt and Pepper” noise with 1% noised pixels



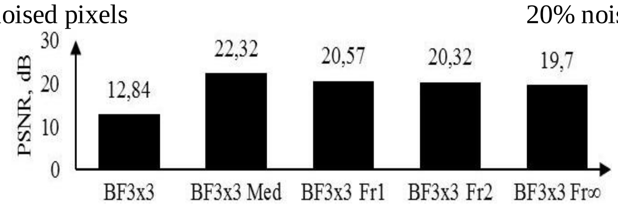
b) Filtration of “Salt and Pepper” noise with 5% noised pixels



c) Filtration of “Salt and Pepper” noise with 10% noised pixels



d) Filtration of “Salt and Pepper” noise with 20% noised pixels

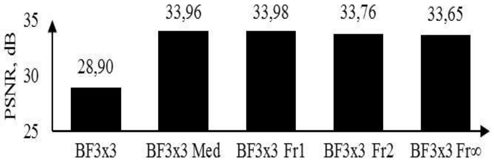


e) Filtration of “Salt and Pepper” noise with 50% noised pixels

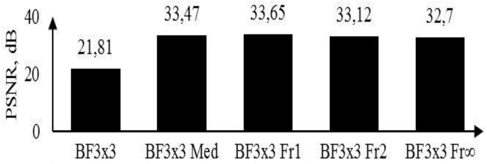
Figure 3. The results for “Salt and Pepper” noise and bilateral filters with Laplacian weights for $\alpha = 0.07$.

Here $\mathbf{min}(x_1, x_2, ..., x_N)$ and $\mathbf{max}(x_1, x_2, ..., x_N)$ are respectively the *minimum* and the *maximum* values among the elements of $(x_1, x_2, ..., x_N)$.

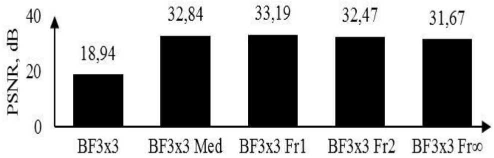
All other properties may come in addition to this fundamental group. For example, if for every permutation $\forall \sigma \in \mathbf{S}_N$ of $\{1, 2, ..., N\}$ the AO satisfies: 4) $\mathbf{Agg}(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(N)}) = \mathbf{Agg}(x_1, x_2, ..., x_N)$, then it is invariant (symmetric) with respect to the permutations of the elements of $(x_1, x_2, ..., x_N)$. In other words, as far as means are concerned, the *order* of the elements of $(x_1, x_2, ..., x_N)$ is - and must be - completely irrelevant.



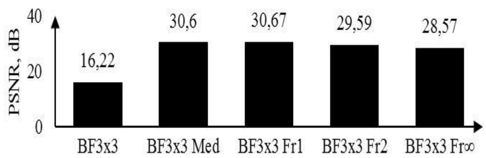
a) Filtration of "Salt and Pepper" noise with 1% noised pixels



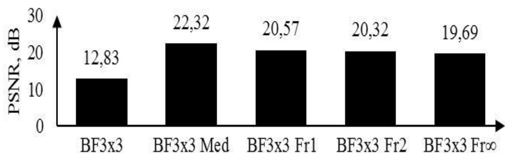
b) Filtration of "Salt and Pepper" noise with 5% noised pixels



c) Filtration of "Salt and Pepper" noise with 10% noised pixels



d) Filtration of "Salt and Pepper" noise with 20% noised pixels



e) Filtration of "Salt and Pepper" noise with 50% noised pixels

Figure 4. The results for "Salt and Pepper" noise and bilateral filters with Laplacian weights for $\alpha = 0.1$.

Proposition 1. (Kolmogorov [13]). If conditions 1)–4) are satisfied, the aggregation $\mathbf{Agg}(x_1, x_2, ..., x_N)$ of the average type are as of the forms:

$$\mathbf{Kolm}\left(K \mid x_1, x_2, ..., x_N\right) = K^{-1}\left[\frac{1}{N}\sum_{i=1}^{N} K\left(x_i\right)\right],$$

where K is a strictly monotone continuous function in the extended real line.

We list below a few particular cases of means:

1) Arithmetic mean ($K(x) = x$): $\mathbf{Mean}(x_1, x_2, ..., x_N) = \frac{1}{N}\sum_{i=1}^{N} x_i$.

2) Geometric mean ($K(x) = \log(x)$): $\mathbf{Geo}(x_1, x_2, ..., x_N) = \exp\left(\frac{1}{N}\sum_{i=1}^{N} \ln x_i\right)$.

3) Harmonic mean ($K(x) = x^{-1}$): $\mathbf{Harm}(x_1, x_2, ..., x_N) = \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i}\right)^{-1}$.

4) A very notable particular case corresponds to the function $K(x) = x^p$. We obtain then a quasi-arithmetic mean of the form: $\mathbf{Power}_p(x_1, x_2, ..., x_N) = \left(\frac{1}{N}\sum_{i=1}^{N} x_i^p\right)^{\frac{1}{p}}$. This family is particularly interesting, because it generalizes a group of common means, only by changing the value of p .A very notable particular cases correspond to the logic functions (min; max; median):

$$y = \mathbf{Min}(x_1, x_2, ..., x_N),\ y = \mathbf{Max}(x_1, x_2, ..., x_N),\ y = \mathbf{Med}(x_1, x_2, ..., x_N).$$

In a 2D standard linear and median scalar filters with a square N -cellular window $M(\mathbf{x})$ and located at $\mathbf{x}$ the mean and median replace the central pixel

$$\hat{s}(\mathbf{x}) = \underset{\mathbf{p}\in M(\mathbf{x})}{\mathbf{Mean}}\left[f(\mathbf{p})\right],\quad \hat{s}(\mathbf{x}) = \underset{\mathbf{p}\in M(\mathbf{x})}{\mathbf{Med}}\left[f(\mathbf{p})\right], \tag{10}$$

where $\hat{s}(\mathbf{x})$ is the filtered grey-level image, $\{f(\mathbf{p})\}_{\mathbf{p}\in M(\mathbf{x})}$ is an image block of the fixed size N extracted from f by moving N -cellular window $M(\mathbf{x})$ at the position $\mathbf{x}$, $\mathbf{Mean}$ and $\mathbf{Med}$ are the mean (average) and median operators. When filters (14) are modified as follows $\hat{s}(\mathbf{x}) = {}^{M}\underset{\mathbf{p}\in M(\mathbf{x})}{\mathbf{Agg}}\left[f(\mathbf{p})\right]$, we get the unique class of nonlinear *aggregation SISO-filters* proposed in [14-16], where ${}^{M}\mathbf{Agg}$ is an aggregation operator on the mask $M(\mathbf{x})$.

For MIMO-filters we have to introduce a vector-valued aggregation. Note, that for ordinary vector-matrix product $\mathbf{g}=\overline{\mathbf{W}}\mathbf{f}$ we have in component-wise form $g_c=\overline{w}_{c1}f_1+\overline{w}_{c2}f_2+\ldots+\overline{w}_{cK}f_K=\sum_{d=1}^{K}\overline{w}_{cd}f_d$.

Let us introduce vector-matrix aggregation product $\mathbf{g}=\overline{\mathbf{W}}_{\mathbf{Agg}}\circ\mathbf{f}$ in component-wise form by the following way $g_c=\mathrm{Agg}\{\overline{w}_{c1}f_1,\overline{w}_{c2}f_2,\ldots,\overline{w}_{cK}f_K,\}=\mathrm{Agg}_{d=1}^{K}\{\overline{w}_{cd}f_d\}$, where Agg is an aggregation operator. Obviously, we can use different aggregation operators in different channels

$$g_c={}^{c}\mathrm{Agg}\{\overline{w}_{c1}f_1,\overline{w}_{c2}f_2,\ldots,\overline{w}_{cK}f_K,\}={}^{c}\mathrm{Agg}_{d=1}^{K}\{\overline{w}_{cd}f_d\}, \quad (15)$$

For $c=1,2,\ldots,K$, where $\mathbf{Agg}=\{{}^{1}\mathrm{Agg},{}^{2}\mathrm{Agg},\ldots,{}^{K}\mathrm{Agg}\}$ is the a K -element set of aggregation operators. In this case we write $\mathbf{g}=\overline{\mathbf{W}}_{\mathbf{Agg}}\circ\mathbf{f}$. When 4-factor MIMO-filter (11) is modified as follows

$$\hat{\mathbf{s}}(\mathbf{x})=\mathbf{MIMO}\,{}^{4}\mathbf{Fact}\,{}^{\mathrm{M}}\mathbf{Agg}_{\mathbf{p}\in M(\mathbf{x})}\left[\overline{\mathbf{W}}_{\mathbf{Agg}}(\mathbf{x},\mathbf{p})\circ\mathbf{f}(\mathbf{p})\right], \quad (16)$$

or in component-wise form

$$\hat{s}_c(\mathbf{x})={}^{M}\mathbf{Agg}_{\mathbf{p}\in M(\mathbf{x})}\left\{{}^{c}\mathbf{Agg}_{d=1}^{K}\left\{\overline{w}^{cd}(\mathbf{x},\mathbf{p})\cdot f_d(\mathbf{p})\right\}\right\} \quad (17)$$

we get the unique class of nonlinear *aggregation MIMO-filters*that we a going to research in future works. They are based on $K+1$ of aggregation operators: 1) ${}^{M}\mathrm{Agg}$ (aggregation on the mask $M(\mathbf{x})$) and 2) $\mathbf{Agg}=\{{}^{1}\mathrm{Agg},{}^{2}\mathrm{Agg},\ldots,{}^{K}\mathrm{Agg}\}$ (inter-channel aggregation), which could be changed independently of one another. For each set of aggregation operators, we get the unique class of new nonlinear filters.

6. Conclusion

A new class of nonlinear generalized 2-, 3-, and 4-factor MIMO-filters for multichannel image processing is introduced in this paper. Weights in 4-factor MIMO-filters include four components: spatial, radiometric, interchannel and interchannel radiometric weights. The fourth weight measures the radiometric distance (for grey-level images) between the interchannel values of the center scalar-valued channel pixel and local neighborhood channel pixels.

7. Acknowledgments

This work was supported by the RFBR grant 19-29-09022\19 and by the Ural State Forest Engineering’s Center of Excellence in «Quantum and Classical Information Technologies for Remote Sensing Systems.

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