Intermediate self-similar asymptotic presentation of the stress and damage fields in the vicinity of the mixed mode crack tip under creep regime

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Abstract. In the paper the class of the creep crack problems in damaged materials under mixed mode loading under creep-damage coupled formulation for plane strain conditions is considered. The class of the asymptotic self-similar solutions to the plane creep crack problems in a damaged medium under mixed-mode loading is given. With the similarity variable and the self-similar representation of the solution for a power-law creeping material and the classical Kachanov – Rabotnov power-law damage evolution equation the near crack-tip stresses, creep strain rates and damage distributions for plane strain and plane stress conditions are obtained. The similarity solutions are based on the idea of the existence of the completely damaged zone near the crack tip. It is shown that the asymptotical analysis of the near crack-tip fields results in nonlinear eigenvalue problems. The technique permitting to find all the eigenvalues numerically is proposed and numerical solutions of the nonlinear eigenvalue problems arising from the mixed-mode crack problems in a power-law medium under plane stress conditions are obtained. Using the approach developed the eigenvalues different from the eigenvalues corresponding to the Hutchinson-Rice-Rosengren (HRR) problem are found. For new eigenspectra and eigensolutions obtained the geometry of the completely damaged zone in the vicinity of the crack tip is found for all values of the mixity parameter.

1. Introduction

Damage accumulation and growth considerations under creep conditions due to the changes in the material microstructure, void nucleation, interaction, and growth on the grain boundaries become very important in the design and operation in such components to ensure structural integrity [1-3]. The basic principles of continuum damage mechanics were initially introduced by Kachanov and Rabotnov [4] to study creep rupture of metal materials with introducing the damage variables which are the average quantity of distributed micro-cracks. Unlike fracture mechanics that can only capture individual crack initiation and propagation, the continuum damage mechanics (CDM) approach captures overall system response with different damage mechanisms and has been extensively used in fracture analysis problems combined with multi failure modes of materials [5].

The HRR [6-8] field is the leading or the first term in an asymptotic analysis of a crack in a power-law hardening material. Though the Hutchinson, Rice and Rosengren (HRR) field [6-8] has been obtained for an ideal discrete crack in intact non-linear hardening materials, the fracture process in usual ductile materials is brought about by nucleation, growth and coalescence of distributed microscopic cavities in front of the crack tip, and this damage field has significant influence on stress field near the crack tip. Therefore, analysis of material damage on the stress and strain fields in the vicinity of the crack tip in nonlinear materials provides very important problems [9-26]. In this context these problems have been discussed in a number of papers [1,2,22,23]. In spite of it systematic information on the effect of material damage on the asymptotic crack-tip field is not available from the analysis. Thus, the paper of Murakami [24] was one of the first work where the effect of damage accumulation process has been elucidated. Asymptotic fields of stress, creep strain rate and damage of a mode I creep crack in steady-state growth are analyzed on the basis of Continuum Damage Mechanics by means of a semi-inverse method. In [25] an asymptotic analysis of the near-tip field is presented in terms of the coordinate perturbation technique for fast crack propagation in an elastic-plastic-viscoplastic materials with damage. A non-singular stress field is obtained, as the damage has substantial influence on the material behavior that the high stresses are relaxed at the crack tip. An analytical expression is obtained which explicitly shows the variation of stresses approaching the crack tip and numerical computations of the angular distributions of stresses and strains are also presented. In [26] in order to evaluate the mechanical behaviour around a growing fatigue crack tip for plane stress of mode I, the asymptotic governing equations and their boundary conditions are formulated by the light of damage mechanics. It is found that the stress has no (or very weak) singularity while the strain is less singular than it is under traditional *K*-dominance. An analytical solution of the nonlinear eigenvalue problem arising from the fatigue crack growth problem in a damaged medium in coupled formulation is obtained in [21]. The perturbation technique for solving the nonlinear eigenvalue problem is used. The method allows us to find the analytical formula expressing the eigenvalue as the function of parameters of the damage evolution law. It is shown that the eigenvalues of the nonlinear eigenvalue problem are fully determined by the exponents of the damage evolution law. In the paper the third - order (four-term) asymptotic expansions of the angular functions determining the stress and continuity fields in the neighborhood of the crack tip are given. The asymptotic expansions of the angular functions permit to find the closed-form solution for the problem considered. In [16] the creep crack problem in damaged materials under mixed mode loading under creep-damage coupled formulation is considered. The class of the self-similar solutions to the plane creep crack problems in a damaged medium under mixed-mode loading is given. With the similarity variable and the self-similar representation of the solution for a power-law creeping material and the power-law damage evolution equation the near crack-tip stresses, creep strain rates and continuity distributions for plane stress conditions are obtained. Using the approach developed the eigenvalues different from the eigenvalues corresponding to the Hutchinson-Rice-Rosengren (HRR) problem are found. Having obtained the eigenspectra and eigensolutions the geometry of the completely damaged zone in the vicinity of the crack tip is found for all values of the mixity parameter.

2. Basic equations

A static mixed mode crack problem under plane strain conditions is considered. The equilibrium equations and compatibility condition in the polar coordinate system can, respectively, be written as

$$r\sigma_{rr,r}+\sigma_{r\theta,\theta}+\sigma_{rr}-\sigma_{\theta\theta}=0,\ \sigma_{\theta\theta,\theta}+r\sigma_{r\theta,r}+2\sigma_{r\theta}=0,\quad 2\left(r\varepsilon_{r\theta,\theta}\right)_{,r}=\varepsilon_{rr,\theta\theta}-r\varepsilon_{rr,r}+r\left(r\varepsilon_{\theta\theta}\right)_{,rr}. \quad (1)$$

The constitutive equations are described by the power law

$$\dot{\varepsilon}_{ij}=\left(3/2\right)B(\sigma_e/\psi)^{n-1}s_{ij}/\psi \quad (2)$$

where s_{ij} are the deviatoric stress tensor components; B,n are material constants; ψ is an integrity (continuity) parameter; $\dot{\varepsilon}_{ij}$ are the creep strain rates. The creep strain rates for plane strain conditions take the form:

$$\dot{\varepsilon}_{rr}=-\dot{\varepsilon}_{\theta\theta}=\frac{3}{4}B\sigma_e^{n-1}\left(\sigma_{rr}-\sigma_{\theta\theta}\right)/\psi^n,\ \dot{\varepsilon}_{r\theta}=\frac{3}{2}B\sigma_e^{n-1}\sigma_{r\theta}/\psi^n,\ \sigma_e=\frac{3}{2}\sqrt{\left(\sigma_{rr}-\sigma_{\theta\theta}\right)^2+4\sigma_{r\theta}^2}. \quad (3)$$

The constitutive model (2) is the phenomenological model of Kachanov and Rabotnov widely employed in creep damage theory and in damage analysis of high temperature structures [27]. The material parameters pertinent to Eqs. 2 for copper, the aluminium alloy, ferritic steels obtained from creep curves are given by Riedel (1987). By noting that the creep damage is brought about by the development of microscopic voids in creep process, L.M. Kachanov represented the damage state by a scalar integrity variable $0 \le \psi \le 1$ where $\psi = 1$ and $\psi = 0$ signify the initial undamaged state and the final completely damaged state (or final fractured state), respectively [9-23]. L.M. Kachanov described the damage development by means of an evolution

$$\dot{\psi} = -A\left(\sigma_e / \psi\right)^m, \tag{4}$$

where $\dot{\psi}$ denotes the time derivative, while A and m are material constants. The solution to Eqs. 1 – 4 should satisfy the traction free boundary conditions on the crack surfaces $\sigma_{\theta\theta}\left(r, \theta = \pm\pi\right) = 0,\ \sigma_{r\theta}\left(r, \theta = \pm\pi\right) = 0.$

The mixed-mode loading can be characterized in terms of the mixity parameter M^p which is defined as

$$M^p = \left(2/\pi\right)\arctan\left|\lim_{r\to 0} \sigma_{\theta\theta}\left(r, \theta = 0\right) / \sigma_{r\theta}\left(r, \theta = 0\right)\right|.$$

The mixity parameter M^p equals 0 for pure mode II; 1 for pure mode I, and $0 < M^p < 1$ for different mixities of modes I and II. Thus, for combine-mode fracture the mixity parameter M^p completely specifies the near-crack-tip fields for a given value of the creep exponent. One can postulate the Airy stress function and the continuity parameter as

$$\chi\left(r, \theta\right) = \sum_{j=0}^{\infty} r^{\lambda_j + 1} f_j(\theta), \quad \psi\left(r, \theta\right) = 1 - \sum_{j=0}^{\infty} r^{\gamma_j + 1} g_j(\theta) \tag{5}$$

First consider the leading terms of the asymptotic expansions (5): $\chi(r,\theta) = r^{\lambda+1} f(\theta)$, $\psi = 1$, where λ is indeterminate exponent and $f(\theta)$ is an indeterminate function of the polar angle, respectively. In view of the asymptotic presentation for the Airy stress potential (5) the asymptotic stress field at the crack tip is derived as follows $\sigma_{ij}(r,\theta) = r^{\lambda-1}\tilde{\sigma}_{ij}(\theta)$, where $\lambda - 1$ denotes the exponent representing the singularity of the stress field will be called the stress singularity exponent hereafter. According to Eq. 2 the asymptotic strain field as $r \to 0$ takes the form $\varepsilon_{ij}(r,\theta) = Br^{(\lambda-1)n}\tilde{\varepsilon}_{ij}(\theta).$ The compatibility condition in Eq. 1 results in the nonlinear forth-order ordinary differential equation (ODE) for the function $f(\theta)$:

$$\begin{gathered}
f_e^2 f^{IV}\left\{(n-1)\left[\left(1-\lambda^2\right)f + f''\right]^2 + f_e^2\right\} - C_2 f_e^4\left[\left(1-\lambda^2\right)f + f''\right] + f_e^4\left(1-\lambda^2\right)f'' + (n-1)(n-3)\left\{\left[\left(1-\lambda^2\right)f + f''\right]\times\right. \\
\left.\times\left[\left(1-\lambda^2\right)f' + f'''\right] + 4\lambda^2 f'f''\right\}^2\left[\left(1-\lambda^2\right)f + f''\right] + (n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f' + f'''\right]^2 + \left[\left(1-\lambda^2\right)f + f''\right]\left(1-\lambda^2\right)f'' +\right. \\
\left. + 4\lambda^2\left(f''^2 + f'f'''\right)\right\}\left[\left(1-\lambda^2\right)f + f''\right] + 2(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f + f''\right]\left[\left(1-\lambda^2\right)f' + f'''\right] + 4\lambda^2 f'f''\right\}\left[\left(1-\lambda^2\right)f' + f'''\right] + \\
+ C_1(n-1)f_e^2\left\{\left[\left(1-\lambda^2\right)f + f''\right]\left[\left(1-\lambda^2\right)f' + f'''\right] + 4\lambda^2 f'f''\right\}f' + C_1 f_e^4 f'' = 0,
\end{gathered} \tag{6}$$

where the following notations

$$f_e^2 = \left[\left(1-\lambda^2\right)f + f''\right]^2 + 4\lambda^2 f'^2,\ C_1 = 4\lambda\left[(\lambda-1)n+1\right], C_2 = (\lambda-1)n\left[(\lambda-1)n+2\right]$$

are adopted. The boundary conditions imposed on the function $f(\theta)$ follow from the traditional traction free boundary conditions on the crack surfaces: $f(\theta = \pm\pi) = 0,\quad f'(\theta = \pm\pi) = 0$.

3. Creep-damage coupled formulation of the problem

In continuum damage mechanics [27] the damage state at an arbitrary point in the material is represented by a properly defined integrity variable $\psi\left(r,\theta\right)$. The integrity parameter reaches its critical value at fracture. According to this notion, a crack in a fracture process can be modeled with the concept of a completely damaged zone in the vicinity of the crack tip. Namely a crack can be represented by a region where the damage state has attained to its critical state $\psi = \psi_{cr}$, i.e., by the completely damaged zone (CDZ). Then the development of the crack and its preceding damage can be elucidated by analyzing the local states of stress, strain and damage. The CDZ may be interpreted as the zone of critical decrease in the effective area due to damage development. Inside the completely damaged zone the damage involved reaches its critical value (for instance, the damage parameter reaches unity) and a complete fracture failure occurs. In view of material damage stresses are relaxed to vanishing [18,19,22,23,27]. Therefore, one can assume that the stress components in the CDZ equal zero. Outside the zone damage alters the stress distribution substantially compared to the corresponding non-damaging material. Well outside the CDZ the continuity parameter is equal to 1. Therefore, asymptotic remote boundary conditions have the form

$$\sigma_{ij}\left(r\to\infty,\theta,t\right)=\left(C^{*}/\left(BI_{n}r\right)\right)^{1/(n+1)}\bar{\sigma}_{ij}(\theta,n) \tag{7}$$

where C^{*} is the path-independent integral, I_n is the constant depending on n. Dimensional analysis of the system formulated shows that the damage mechanics equations must have similarity solutions of the form

$$\sigma_{ij}(r,\theta,t)=\left(At\right)^{-1/m}\hat{\sigma}_{ij}(R,\theta),\ \psi(r,\theta,t)=\hat{\psi}(R,\theta)$$

where $R=r\left(At\right)^{-(n+1)/m}BI_{n}/C^{*}$ is the similarity variable. It should be noted that the remote boundary conditions can be formulated in a more general form compared with Eq. 7 $\sigma_{ij}\left(r\to\infty,\theta,t\right)=C_{1}r^{s}\bar{\sigma}_{ij}(\theta,n)$, where the stress singularity exponent s is unknown and has to be determined as a part of solution, C_1 is the amplitude of the stress field at infinity defined by the specimen configuration and loading conditions. For the power-law constitutive relations, the power damage evolution law (4) and the more general remote boundary conditions the self-similar variable $R=r\left(AtC_{1}^{m}\right)^{1/(sm)}$ can be introduced. The asymptotic solution outside the completely damaged zone is sought in the form

$$\chi(R,\theta)=\sum\nolimits_{j=0}^{\infty}R^{\lambda_j+1}f_j(\theta),\qquad \psi(R,\theta)=1-\sum\nolimits_{j=0}^{\infty}R^{\gamma_j}g_j(\theta).$$

4. The geometry of the totally damaged zone in vicinity of the mixed mode crack tip

The asymptotic approach developed here allows us to find the geometry of the completely damaged zone in the vicinity of the crack tip for different values of the mixity parameter and material constants. The configurations of the completely damaged zones encompassing the crack tip are shown in figures. 1-3. Altenbach, Matsuda and Okumura [9] present numerical analysis of a crack-tip field in particulate-reinforced composites with debonding damage and containing various sized particles has been carried out by the FEM. FEM analysis was carried out for the three kinds of composites in the case of no debonding damage (perfect composite) and with debonding damage (composite with damage). On the composites with damage, with increasing particle size, the debonding damage becomes easy to occur and damage zone spreads out widely and a result the macroscopic equivalent stress is drastically reduced. The geometry of the active damage accumulation zones given by Altenbach, Matsuda and Okumura [9] is very similar to the configurations shown in figure. 3. Thus, on can conclude that the asymptotic analysis presented here allows us to obtain new asymptotic behaviour of the stress filed in the vicinity of the crack tip and take into account the damage evolution process.

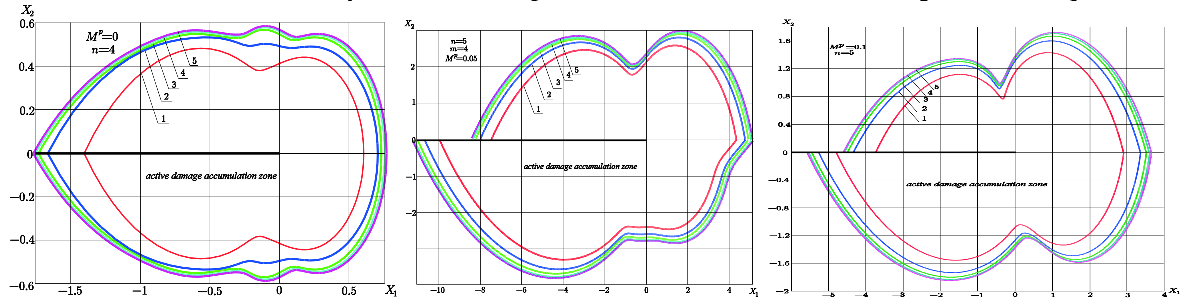

Figure 1. The geometry of the completely damaged zone in the vicinity of the crack tip.

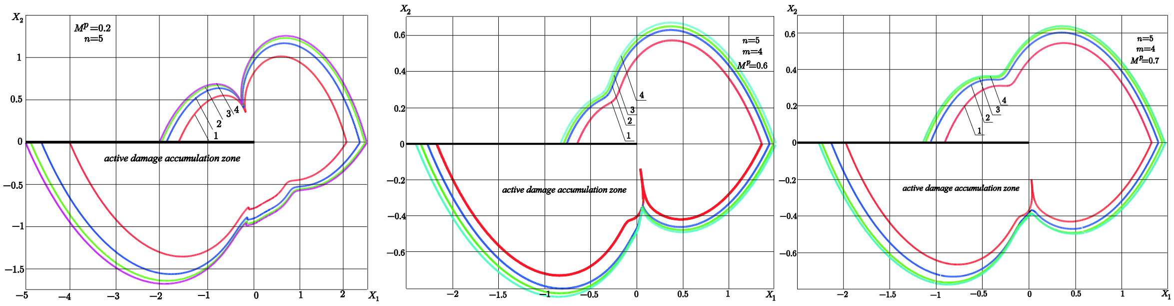

Figure 2. The geometry of the completely damaged zone in the vicinity of the crack tip under mixed mode loading.

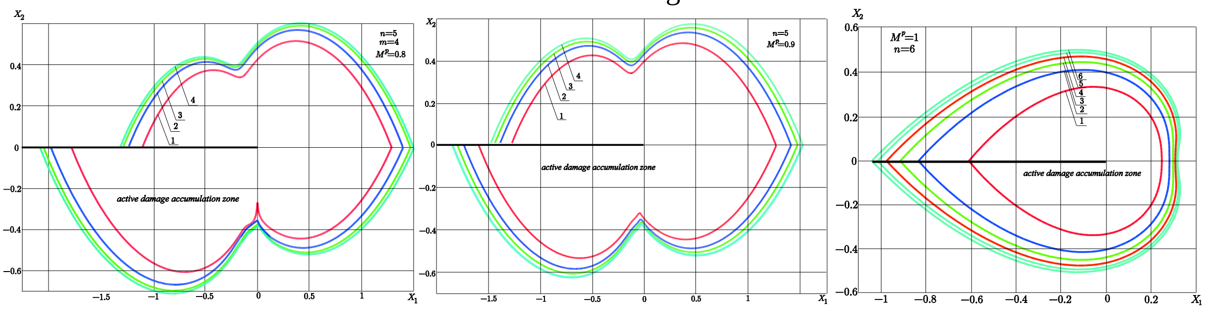

Figure 3. The geometry of the completely damaged zone in the vicinity of the crack tip under mixed mode loading.

5. Conclusions

The paper presents the asymptotic analysis of the stress-strain state in the vicinity of the crack tip in damaged materials. The geometry of the completely damaged zone in the vicinity of Mixed Mode I/II crack tip for all range of the values of the mixity parameter is studied. It is shown that new asymptotic behaviour of the stresses governs the damage accumulation zone.

6. References

[1] Shlyannikov, V. Creep damage and stress intensity factor assessment for plane multi-axial and three-dimensional problems / V. Shlyannikov, A. Tumanov // International Journal of Solids and Structures. – 2018. – Vol. 150. – P. 1-18.

[2] Shlyannikov, V. Creep-fatigue crack growth rate assessment using ductility damage model / V. Shlyannikov, A. Tumanov, N. Boychenko // International Journal of Fatigue. – 2018. – Vol. 116. – P. 448-461.

[3] Gross, D. Fracture mechanics: Introduction to Micromechanics / D. Gross, T. Seeling. – Berlin: Springer, 2018. – 366 p.

[4] Rabotnov, Y.N. Creep Problems in Structure Members / Y.N. Rabotnov – Amsterdam : North-Holland, 1969. – 600 p.

[5] Yun, K. A damage model based on the introduction of a crack direction parameter for FPR composites under quasi-static load / K. Yun, Z. Wang, L. He, J. Liu // Composite structures. – 2018. – Vol.184. – P. 388-399.

[6] Hutchinson, J.W. Singular behaviour at the end of a tensile crack in a hardening material / J.W. Hutchinson // J Mech Phys Solids. – 1968. – Vol. 16. – P. 13-31.

[7] Hutchinson, J.W. Plastic stress and strain fields at a crack tip / J.W. Hutchinson // J Mech Phys Solids. – 1968. – Vol. 16. – P. 337-347.

[8] Rice, J.R. Plain strain deformation near a crack tip in a power-law hardening material / J.R. Rice, G.F. Rosengren // J Mech Phys Solids. – 1968. – Vol. 16. – P. 1-12.

[9] Altenbach, H. From Creep Damage Mechanics to Homogenezation Methods / H. Altenbach, T. Matsuda, D. Okumura. – Berlin: Springer, 2015. – 601 p.

[10] Altenbach, H. Failure and Damage Analyses of Advanced Materials / H. Altenbach, T. Sadowski. – Berlin: Springer, 2015. – 278 p.

[14] Barenblatt, G.I. Flow, Deformation and Fracture: Lectures on Fluid Mechanics and the Mechanics of Deformable Solids for Mathematicians and Physicists / G.I. Barenblatt. – Cambridge: Cambridge University Press, 2014. – 276 p.
[15] Bui, H.D. Fracture Mechanics. Inverse Problems and Solutions / H.D. Bui. – Dordrecht: Springer, 2006. – 394 p.
[16] Stepanova, L.V. Stress-strain state in the vicinity of a crack under mixed loading / L.V. Stepanova, E.M. Adylina // Journal of Applied Mechanics and Technical Physics. – 2014. – Vol. 55(5). – P. 885-895.
[17] Chousal, J.A.G. Mixed mode I+II continuum damage model applied to fracture characterization of bonded joints / J.A.G. Chousal, M.F.S.F. de Moura // Int. J. of Adhesion and Adhesives. – 2013. – Vol. 41. – P. 92-97.
[18] Stepanova, L.V. Eigenvalue analysis for a crack in a power-law material / L.V. Stepanova // Computational Mathematics and Mathematical Physics. – 2009. – Vol. 49(8). – P. 1332-1347.
[19] Stepanova, L.V. Mixed-mode loading of the cracked plate under plane stress conditions / L.V. Stepanova, E.M. Yakovleva // PNRPU Mechanics Bulletin. – 2014. – Vol. 3. – P. 129-162.
[20] Kuna, M. Finite Elements in Fracture Mechanics. Theory-Numerics-Applications / M. Kuna. – Dordrecht: Springer, 2013. – 447 p.
[21] Stepanova, L.V. Perturbation method for solving the nonlinear eigenvalue problem arising from fatigue crack growth problem in a damaged medium / L.V. Stepanova, S.A. Igonin // Applied Mathematical Modelling. – 2014. – Vol. 38(14). – P. 3436-3455.
[22] Stepanova, L.V. Eigenspectra and orders of stress singularity at a mode I crack tip for a power-law medium. Comptes Rendus. Mecanique. – 2008. – Vol. 336(1-2). – P. 232-237.
[23] Stepanova, L. Eigenvalues of antiplane-shear crack problem for a power-law material / L. Stepanova // Journal of Applied Mechanics and Technical Physics. – 2008. – Vol. 49(1). – P. 142-147.
[24] Murakami, S. Asymptotic fields of stress and damage of a mode I creep crack in steady – state growth / S. Murakami, T. Hirano, Y. Liu // International Journal of Solids and Structures. – 2000. – Vol. 37. – P. 6203-6220.
[25] Lu, M. Crack-tip field for fast fracture of an elastic-plastic-viscoplastic material incorporated with quasi-brittle damage / M. Lu, Y.W. Mai, L. Ye // International Journal of Solids and Structures. – 2001. – Vol. 38. – P. 9383-9402.
[26] Zhao, J. The asymptotic study of fatigue crack growth based on damage mechanics / J. Zhao, X. Zhang // Engineering Fracture Mechanics. – 1995. – Vol. 50(1). – P. 131-141.
[27] Murakami, S. Continuum Damage Mechanics. A Continuum Mechanics Approach to the Analysis of Damage and Fracture / S. Murakami. – Dordrecht: Springer, 2012. – 600 p.

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