Decomposition of travelling wave existence problem for singularly perturbed semilinear parabolic equations

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Abstract. The travelling waves problem for the singularly perturbed semilinear parabolic equations is considered in the paper.It is shown that the corresponding problem for a singularly perturbed ODE system can be reduced to a certain problem of lower dimension using a splitting transformation based on the technique of slow and fast integral manifolds.

1. Introduction

Travelling waves play a fundamental role in many mathematical equations, see, for example, [1, 2, 3]. We can write a number models of interest in the general form

$$\frac{\partial u}{\partial t} = \varepsilon D \frac{\partial^2 u}{\partial x^2} + F(u), \tag{1}$$

$u \in R^n$, $x \in R$, ≥ 0, ε is a small positive parameter. Here, D is the constant diagonal matrix $D = diag(D_1, D_2, ..., D_n)$, $F(u)$ is the sufficiently smooth and bounded vector-function.

Equations of these types are widely used as mathematical models of biology, chemistry and physics, and many examples in phenomena resembling travelling waves have been found empirically.

A travelling wave is a solution which is of the form $u(x;t) = u(\xi)$ with $\xi = x - st$ for some wave speed $s \in R$ and which satisfies the following ODEs:

$$-su' = \varepsilon D u'' + P(u), \tag{2}$$

where $()'$ refers to differentiation with respect to ξ.

Our main goal is the reduction of (2) to the equation of lower order using the splitting transformation [4, 5].

2. Splitting transformation

Consider the differential system that is linear with respect to fast variables

$$\dot{x} = f(x,t,\varepsilon) + F(x,t,\varepsilon)y, \tag{3}$$

$$\varepsilon \dot{y} = g(x,t,\varepsilon) + G(x,t,\varepsilon)y, \tag{4}$$

where $x \in R^m,\ y \in R^n,\ t \in R$. We assume that the eigenvalues $\lambda_i(x,t)$ of the matrix $G(x,t,0)$ have the property $Re\lambda_i(x,t) \le -2\gamma < 0$, in $t \in \mathbb{R}, x \in \mathbb{R}^m$, and that the matrix- and vector-functions f, g, F and G are continuous and bounded as well as their partial derivatives with respect to the arguments $t \in \mathbb{R}, x \in \mathbb{R}^m, \varepsilon \in [0, \varepsilon_0]$.

When these assumptions hold, the system (3)-(4) has a slow integral manifold

$$y = h(x,t,\varepsilon) = h_0(x,t) + \varepsilon h_1(x,t) + \dots .$$

Noting that

$$\frac{dy}{dt} = \frac{\partial h}{\partial t} + \frac{\partial h}{\partial x}(f + Fh),$$

on using the first of (3)-(4)), the functions h_i can be derived from the invariance equation

$$\varepsilon\frac{\partial h}{\partial t} + \varepsilon\frac{\partial h}{\partial x}(f + Fh) = g + Gh.$$

Suppose we have the following representations

$$F(x,t,\varepsilon) = \sum_{j\ge 0}\varepsilon^j F_j(x,t), \quad G(x,t,\varepsilon) = \sum_{j\ge 0}\varepsilon^j G_j(x,t),$$

$$f(x,t,\varepsilon) = \sum_{j\ge 0}\varepsilon^j f_j(x,t), \quad g(x,t,\varepsilon) = \sum_{j\ge 0}\varepsilon^j \xi j(x,t).$$

Here, $G_0 = G_0(x,t)$ plays the role of the matrix $B(x,t)$. The formulae for the coefficients of the asymptotic expansions of slow integral manifold $h = h(x,t,\varepsilon)$ take the form

$$\begin{aligned} h_0 &= G_0^{-1}\xi_0, \\ h_k &= G_0^{-1}[\frac{\partial h_{k-1}}{\partial t} + \sum_{p=0}^{k-1}\frac{\partial h_p}{\partial x}(f_{k-1-p} + \sum_{j=0}^{k-1-p}F_j h_{k-p-1-j}) \\ &- \xi_k - \sum_{j=1}^{k} G_j h_{k-j}], \quad k \ge 1. \end{aligned} \tag{5}$$

The invariance equation for the fast integral manifold $H = H(v,z,t,\varepsilon)$ [4, 5] takes the form

$$\varepsilon\frac{\partial H}{\partial t} + \varepsilon\frac{\partial H}{\partial v}[f(v,t,\varepsilon) + F(v,t,\varepsilon)h(v,t,\varepsilon)] + \frac{\partial H}{\partial z}[G(v+\varepsilon H,t,\varepsilon)$$

$$-\varepsilon\frac{\partial h}{\partial x}(v+\varepsilon H,t,\varepsilon)F(v+\varepsilon H,t,\varepsilon)]z = f(v+\varepsilon H,t,\varepsilon) - f(v,t,\varepsilon)$$
$$+F(v+\varepsilon H,t,\varepsilon)(z + h(v+\varepsilon H,t,\varepsilon)) - F(v,t,\varepsilon)h(v,t,\varepsilon).$$

Setting $\varepsilon = 0$, we obtain

$$\frac{\partial H_0}{\partial z}G_0(v,t)z = F_0(v,t)z.$$

It is possible to represent $H_0(v,t,z)$ in the form $H_0(v,z,t) = D_0(v,t)z$, where the matrix $D_0(v,t)$ satisfies the equation

$$D_0(v,t)G_0(v,t) = F_0(v,t),$$

and, therefore,

$$H_0(v,z,t) = F_0(v,t)G_0^{-1}(v,t)z.$$

Our main goal is the construction of the transformation

$$x = v + \varepsilon H(v, z, t, \varepsilon), \tag{6}$$

$$y = z + h(x, t, \varepsilon), \tag{7}$$

which reduces the original system (3), (4) to the form

$$\dot{v} = \varphi(v, t, \varepsilon), \tag{8}$$

$$\varepsilon \dot{z} = \eta(v, z, t, \varepsilon). \tag{9}$$

Let $(x(t),\ y(t))$ be a solution to (3), (4) with an initial condition $x(t_0) = x_0,\ y(t_0) = y_0$. There exists a solution $(v(t),\ z(t))$ of (8), (9) with the initial condition $v(t_0) = v_0,\ z(t_0) = z_0$, such that

$$x(t) = v(t) + \varepsilon H(v(t), z(t), t, \varepsilon), \quad y(t) = z(t) + h(x(t), t, \varepsilon). \tag{10}$$

It is sufficient to show that (10) holds for $t = t_0$. Setting $t = t_0$ in (10) we obtain

$$x_0 = v_0 + \varepsilon H(v_0, z_0, t_0, \varepsilon), \quad y_0 = z_0 + h(x_0, t_0, \varepsilon),$$

and, therefore, $z_0 = y_0 - h(x_0, t_0, \varepsilon)$.

For v_0 we have the equation

$$v_0 = x_0 - H(v_0, z_0, t_0, \varepsilon) = V(v_0), \tag{11}$$

which has a unique solution for any $x_0 \in R^m$ and fixed z_0 and t_0, where

$$\|z_0\| = \|y_0 - h(x_0, t_0, \varepsilon)\| \leq \rho_1$$

for some ρ_1.

The following statement is true. Any solution $x = x(t, \varepsilon),\ y = y(t, \varepsilon)$ of system (3), (4) with the initial condition $x(t_0, \varepsilon) = x_0,\ y(t_0, \varepsilon) = y_0$, can be represented in form of (10).

This statement means that (3), (4) can be reduced to the form (8), (9) by the splitting transformation (6), (7). Thus, system (3), (4) was split into two subsystems, the first of which is independent and contains a small parameter in a regular manner. Note that the initial value v_0 can be calculated from (11) in the form of an asymptotic expansion:

$$v_0 = v_{00} + \varepsilon v_{01} + \varepsilon^2 v_{02} + \dots\ .$$

For example, $v_{00} = z_0$, $v_{01} = -H(x_0, z_{00}, t_0, 0)$, where $z_{00} = y_0 - h(x_0, t_0)$.

It is important to underline that there exists number $K,\ \ K > 1$ such that

$$\|z(t, \varepsilon)\| \leq K \exp(-\gamma t/\varepsilon) \|z_0\|, \quad t \geq 0.$$

This means: the solution $x = x(t, \varepsilon)$, $y = y(t, \varepsilon)$ of the original system (3)-(4) that satisfied the initial condition $x(0, \varepsilon) = x_0$, $y(t_0, \varepsilon) = y_0$ can be represented as

$$x(t, \varepsilon) = v(t, \varepsilon) + \varepsilon \varphi_1(t, \varepsilon),$$

$$y(t, \varepsilon) = \bar{y}(t, \varepsilon) + \varphi_2(t, \varepsilon).$$

Thus, this solution is represented as a sum of solution which lies on the slow integral manifold, i.e.

$$x = x(t, \varepsilon) = v(t, \varepsilon), \quad y(t, \varepsilon) = h(v(t, \varepsilon), t, \varepsilon),$$

and exponentially decreasing functions

$$\varepsilon\varphi_1(t,\varepsilon) = \varepsilon H(v(t,\varepsilon), z(t,\varepsilon), t, \varepsilon),$$

$$\varphi_2(t,\varepsilon) = z(t,\varepsilon) + h(v(t,\varepsilon) + \varepsilon H(v(t,\varepsilon), z(t,\varepsilon), t, \varepsilon), t, \varepsilon) - h(v(t,\varepsilon), t, \varepsilon).$$

Neglecting terms of order $o(\varepsilon)$, we use the transformation

$$x = v + \varepsilon H_0(v,z,t), \quad y = z + h_0(x,t) + \varepsilon h_1(x,t)$$

to reduce system (3)- (4) to a nonlinear block-triangular form:

$$\dot{v} = f_0(v,t) + F_0(v,t)h_0(v,t) + \varepsilon[f_1(v,t)$$
$$+F_0(v,t)h_1(x,t) + F_1(v,t)h_0(v,t)] + O(\varepsilon^2),$$
$$\varepsilon\dot{z} = [G_0(v,t) + \varepsilon(G_1(v,t) + \frac{\partial G_0}{\partial x}(v,t)H_0(v,z,t)$$
$$-\frac{\partial h_0}{\partial x}(v,t)F_0(v,t))]z + O(\varepsilon^2).$$

3. Travelling wave problem decomposition

Suppose that the speed of the travelling wave is a value of order unity, i.e. $s = O(1)$ as $\varepsilon \to \infty$. It is possible to rewrite (2) of form (3)-(4) with

$$f = 0, \quad F = I, \quad g = -D^{-1}P, \quad G = -sD^{-1}.$$

These formulae and (5) imply

$$\dot{v} = h_0(v) + \varepsilon h_1(v) + O(\varepsilon^2), \tag{12}$$

where

$$h_0 = G^{-1}g = \frac{1}{s}P(v)$$

and

$$h_1 = G^{-1}\frac{\partial h_0(v)}{\partial v}h_0(v).$$

Thus, if we find the periodic solution of (12) then we have the periodic travelling wave to the original equation (1). The situation is similar in the case of a heteroclinic or homoclinic trajectory of the equation (12).

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5. References

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