Asymptotic Methods and Their Applications in Nonlinear Fracture Mechanics

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Abstract. Nowadays important advances in mathematical models of nonlinear fracture mechanics have been made in the last two decades as scientists and engineers strive to imbue mathematical models with more realistic details at microstructure fracture and damage mechanisms in the failure process. Such mathematical models are reviewed with the aim of providing users insight into the key ideas, features and differences of prevailing models in this paper. Several models and asymptotic solutions are described. Some applications for these models are also given.

Keywords: perturbation theory, asymptotic methods, crack tip fields, stress singularity, nonlinear eigenvalue problem, mixed mode loading, power law materials.

1. Introduction

Knowledge of stress, strain and displacement fields in the vicinity of the crack tip under mixed-mode loading conditions is important for the justification of fracture mechanics criteria and has attracted considerable attention nowadays [1-9]. Damage field around a crack tip essentially affects the surrounding stress field, and hence governs the crack extension behaviour in the material. This effect of the damage field is an important problem either in the discussion of stability and convergence in crack extension analysis. So far mainly crack problems for the pure opening mode I at symmetrical loading have been thoroughly treated [1-3]. The corresponding fracture criteria have been obtained on the assumption that the crack continues to extend along its original line (two-dimensional case) or plane (three-dimensional case) in a straightforward manner on the ligament. Nowadays the analysis of mixed-mode loading of cracked structures in nonlinear materials is of particular interest. In engineering practice, there are plenty of examples and reasons leading to mixed-mode loading of cracked structures when mode I is superimposed by mode II and/or III, the symmetry (or antisymmetry) is violated and the situation is related to mixed-mode loading [4]. The type of loading on a structure (tension, shear, bending, torsion) can also change during service. For a crack this results in an alteration of opening mode I, II and III which is why the study of mixed-mode loads is of particular importance [1-6]. In linear fracture mechanics the principle of superposition allows to obtain solutions for mixed mode I/II crack problems whereas in nonlinear fracture mechanics many questions are still open [8,10-12]. Analysis of the near crack-tip fields in power-law hardening (or power-law creeping) damaged materials under mixed-mode loading results in new nonlinear eigenvalue problems in which the whole spectrum of the eigenvalues and orders of stress singularity have to be determined

[10-12]. The objective of this study is to analyze the crack-tip fields in a damaged material under mixed-mode loading conditions and to consider the meso-mechanical effect of damage on the stress-strain state near the crack tip.

2. Mathematical formulation of the problem and basic equations

A static mixed mode crack problem under plane stress conditions is considered. The equilibrium equations and compatibility condition in the polar coordinate system can, respectively, be written as

$$r\sigma_{rr,r}+\sigma_{r\theta,\theta}+\sigma_{rr}-\sigma_{\theta\theta}=0,\ \sigma_{\theta\theta,\theta}+r\sigma_{r\theta,r}+2\sigma_{r\theta}=0,$$

$$2(r\varepsilon_{r\theta,\theta})_{,r}=\varepsilon_{rr,\theta\theta}-r\varepsilon_{rr,r}+r(r\varepsilon_{\theta\theta})_{,rr}.$$

The constitutive equations are described by the power law $\dot{\varepsilon}_{ij}=(3/2)B(\sigma_e^{n-1}/\psi)s_{ij}/\psi$, where s_{ij} are the deviatoric stress tensor components; B,n are material constants; ψ is an integrity (continuity) parameter; $\dot{\varepsilon}_{ij}$ are the strain components which for the plane stress conditions take the form: $\dot{\varepsilon}_{rr}=B\sigma_e^{n-1}\left(2\sigma_{rr}-\sigma_{\theta\theta}\right)/\left(2\psi^n\right),\ \dot{\varepsilon}_{\theta\theta}=B\sigma_e^{n-1}\left(2\sigma_{\theta\theta}-\sigma_{rr}\right)/\left(2\psi^n\right),\ \dot{\varepsilon}_{r\theta}=3B\sigma_e^{n-1}\sigma_{r\theta}/\left(2\psi^n\right).$

The Mises equivalent stress is expressed by $\sigma_e=\sqrt{\sigma_{rr}^2+\sigma_{\theta\theta}^2-\sigma_{rr}\sigma_{\theta\theta}+3\sigma_{r\theta}^2}$.

The constitutive model described before is the phenomenological model of Kachanov and Rabotnov widely employed in creep damage theory and in damage analysis of high temperature structures [3,4,8,10,13,14]. The material parameters pertinent to power-law creeping materils for copper, the aluminium alloy, ferritic steels obtained from creep curves are given in [15]. By noting that the creep damage is brought about by the development of microscopic voids in creep process, L.M. Kachanov [13-16] represented the damage state by a scalar integrity variable $\psi\ \left(0\le\psi\le1\right)$ where $\psi=1$ and $\psi=0$ signify the initial undamaged state and the final completely damaged state (or final fractured state), respectively. L.M. Kachanov described the damage development by means of an evolution equation $\dot{\psi}=-A\left(\sigma_e/\psi\right)^m$, where $\dot{\psi}$ denotes the time derivative, while A and m are material constants. The solution of the system formulated should satisfy the traditional traction free boundary conditions on the crack surfaces $\sigma_{\theta\theta}\left(r,\theta=\pm\pi\right)=0,\ \sigma_{r\theta}\left(r,\theta=\pm\pi\right)=0.$

The mixed-mode loading can be characterized in terms of the mixity parameter M^p which is defined as $M^p=\left(2/\pi\right)\arctan\left|\lim\limits_{r\to0}\sigma_{\theta\theta}\left(r,\theta=0\right)/\sigma_{r\theta}\left(r,\theta=0\right)\right|.$

The mixity parameter M^p equals 0 for pure mode II; 1 for pure mode I, and $0<M^p<1$ for different mixities of modes I and II. Thus, for combine-mode fracture the mixity parameter M^p completely specifies the near-crack-tip fields for a given value of the creep exponent. By postulating the Airy stress function $\chi\left(r,\theta\right)$ expressed in the polar coordinate system, the stress components state are expressed as: $\sigma_{\theta\theta}=\chi_{,rr}$, $\sigma_{rr}=\chi_{,r}/r+\chi_{,\theta\theta}/r^2$, $\sigma_{r\theta}=-\left(\chi_{,\theta}/r\right)_{,r}$. As for the asymptotic stress field at the crack tip $r\to0$, one can postulate the Airy stress function and the continuity parameter as

$$\chi\left(r,\theta\right)=\sum_{j=0}^{\infty}r^{\lambda_j+1}f_j(\theta),\quad \psi\left(r,\theta\right)=1-\sum_{j=0}^{\infty}r^{\gamma_j+1}g_j(\theta).$$

First consider the leading terms of the asymptotic expansions for the integrity parameter: $\chi(r,\theta)=r^{\lambda+1}f(\theta)$, $\psi=1$, where λ is indeterminate exponent and $f(\theta)$ is an indeterminate function of the polar angle, respectively. In view of the asymptotic presentation for the Airy stress potential the asymptotic stress field at the crack tip is derived as follows $\sigma_{ij}(r,\theta)=r^{\lambda-1}\tilde{\sigma}_{ij}(\theta)$, where $\lambda-1$ denotes the exponent representing the singularity of the stress field, and will be called the stress singularity exponent hereafter. The asymptotic strain field as $r\to0$ takes the form $\varepsilon_{ij}(r,\theta)=Br^{(\lambda-1)n}\tilde{\varepsilon}_{ij}(\theta).$ The compatibility condition results in the nonlinear forth-order ordinary differential equation (ODE) for the function $f(\theta)$:

$$f^{IV} f_e^2 \left\{(n-1)\left[(\lambda+1)(2-\lambda) f+2 f''\right]^2/2+2 f_e^2\right\}+6\left[(\lambda-1)n+1\right]\lambda\left\{(n-1) f_e^2 h f'+f_e^4 f''\right\}+(n-1)(n-3)h^2 \times$$

$$\times\left[(\lambda+1)(2-\lambda) f+2 f''\right]+(n-1) f_e^2\left[(\lambda+1)(\lambda+2) f+2 f''\right]\left\{\left[(\lambda+1) f'+f'''\right]^2+\left[(\lambda+1) f+f''\right](\lambda+1) f''+\right.$$

$$+(\lambda+1)^2\lambda^2\left(f'^2+ff''\right)-(\lambda+1)^2\lambda ff''/2-\left[(\lambda+1) f'+f'''\right](\lambda+1)\lambda f'+f_e^4(\lambda+1)(2-\lambda) f''-$$

$$\left.-\left[(\lambda+1) f+f''\right](\lambda+1)\lambda f''+3\lambda^2\left(f''^2+f'f'''\right)\right\}+2(n-1) f_e^2 h\left[(\lambda+1)(2-\lambda) f'+2 f'''\right]/2-$$

$$-(\lambda-1)n f_e^4\left[(\lambda+1)(2-\lambda) f+2 f''\right]+\left[(\lambda-1)n+1\right](\lambda-1)n f_e^4\left[(\lambda+1)(2\lambda-1) f-f''\right]=0,$$

where the following notations are adopted

$$f_e=\sqrt{\left[(\lambda+1) f+f''\right]^2+(\lambda+1)^2\lambda^2 f^2-\left[(\lambda+1) f+f''\right](\lambda+1)\lambda f+3\lambda^2 f'^2},\quad h=\left[(\lambda+1) f+f''\right]\times$$

$$\times\left[(\lambda+1) f'+f'''\right]+(\lambda+1)^2\lambda^2 ff'-\left[(\lambda+1) f'+f'''\right](\lambda+1)\lambda f/2-\left[(\lambda+1) f+f''\right](\lambda+1)\lambda f'/2+3\lambda^2 f'f''.$$

Thus the eigenfunction expansion method results in the nonlinear eigenvalue problem: it is necessary to find eigenvalues λ leading to nontrivial solutions of the nonlinear differential equation obtained satisfying the boundary conditions. Therefore, the order of the stress singularity is the eigenvalue and the angular variations of the field quantities correspond to the eigenfunctions. When we consider mode I loading or mode II loading conditions symmetry or antisymmetry requirements of the problem with respect to the crack plane at $\theta=0$ are utilized. Due to the symmetry (or antisymmetry) the solution is sought for one of the half-planes. In analyzing the crack problem under mixed-mode loading conditions the symmetry or antisymmetry arguments can not be used and it is necessary to seek for the solution in the whole plane $-\pi\le\theta\le\pi$. To find the numerical solution one has to take into account the value of the mixity parameter. For this purpose in the framework of the proposed technique the nonlinear ordinary differential equation obtained is numerically solved on the interval $[0,\pi]$ and the two-point boundary value problem is reduced to the initial problem with the initial conditions reflecting the value of the mixity parameter

$$f(\theta=0)=1,\; f'(\theta=0)=(\lambda+1)/tg\left(M^p\pi/2\right),\quad f(\theta=\pi)=0,\quad f'(\theta=\pi)=0.$$

The first initial condition is the normalization condition. The second condition follows from the value of the mixity parameter specified. At the next step the numerical solution of Eq. 6 is found on the interval $[-\pi,0]$ with the following boundary conditions

$$f(\theta=-\pi)=0,\; f'(\theta=-\pi)=0,\quad f(\theta=0)=1,\quad f'(\theta=0)=(\lambda+1)/tg\left(M^p\pi/2\right).$$

3. Numeric solution

The analogous approach has been realized in [8] where the near mixed-mode crack-tip stress field under plane strain conditions was analyzed. It is assumed that the eigenvalue of the problem considered equals the eigenvalue of the classical HRR problem $\lambda=n/(n+1)$. However, it turns out that when we construct the numerical solution for the mixed-mode crack problem the radial stress component $\sigma_{rr}(r,\theta)$ has discontinuity at $\theta=0$ whereas for the cases of pure mode I and pure mode II loadings when $M^p=1$ and $M^p=0$ are valid the radial stress component is continuous at $\theta=0$. Numerical analysis carried out previously for mixed-mode crack problem under plane strain conditions leads to the continuous angular distributions of the radial stress component $\sigma_{rr}(r,\theta=0)$ [8]. Thus one can compute the whole set of eigenvalues for plane stress conditions from the continuity requirements of the radial stress components on the line extending the crack. In accordance with the procedure proposed the spectrum of the eigenvalues λ is numerically obtained. Results of computations are shown in Table 1 where the new eigenvalues λ computed and the values of the functions $f''(\theta=0)$, $f'''(\theta=0)$, $f''(\theta=-\pi)$ and $f'''(\theta=-\pi)$ numerically obtained for the different values of the mixity parameter M^p and the creep exponent n are given. The angular distributions of the stress components for different values of creep exponent n and for all values of the mixity parameter M^p are shown in Fig. 1.

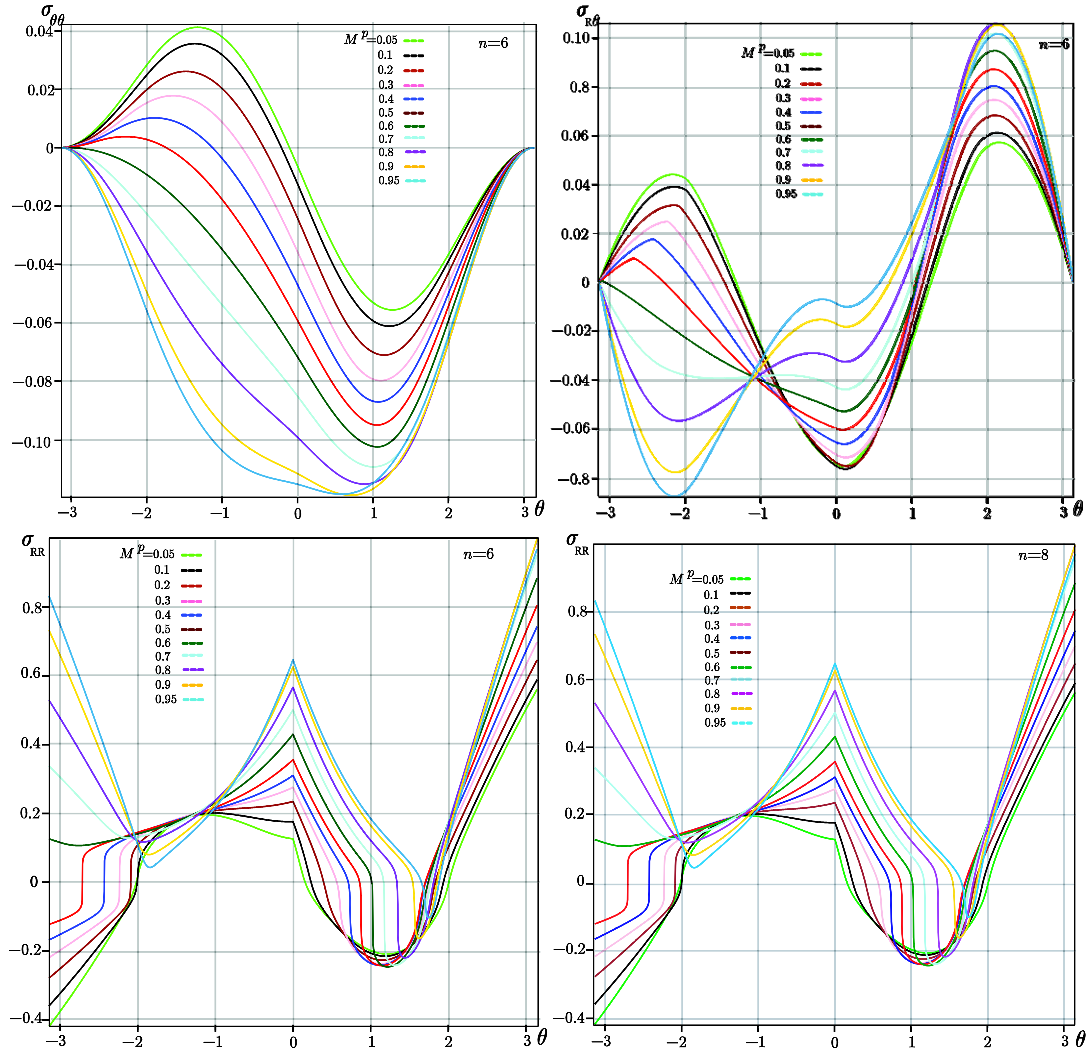

Figure 1. Angular distributions of the stress components for different values of the mixity parameter.

Table 1. Eigenvalues for different values of mixity parameter for plane stress onditions $n = 2$

M^p	λ	$f''(\theta=0)$	$f'''(\theta=0)$	$f''(\theta=-\pi)$	$f'''(\theta=-\pi)$
0.9	-0.30032000	-0.25428500	-0.52319280	0.36781000	0.41793000
0.8	-0.28609000	0.30988600	-0.65543910	-0.14222000	1.23657500
0.7	-0.26789000	-0.40297913	-0.80444475	-0.37921000	0.54939150
0.6	-0.26093000	-0.46493199	1.03847110	-0.54340000	0.46094200
0.5	-0.25233200	-0.52217930	-1.40075019	-0.72780000	0.42459230
0.4	-0.24369800	-0.57136233	-1.98711539	-0.97155000	0.40989380
0.3	-0.23701900	-0.61089207	-2.95625279	-1.35116000	0.41294900
0.2	-0.23247900	-0.64000914	-4.76598544	-2.08610169	0.44422935
0.1	-0.22987230	-0.65774480	-9.82544937	-4.26300089	0.57184713

The method proposed has been applied to nonlinear eigenvalue problems arising from the problem of the determining the near crack-tip fields in the damaged materials. In continuum damage mechanics

(Altenbach and Sadowski (2015), Murakami (2012), Kuna (2013), Voyiadis (2015), Voyiadis and Kattan (2012), Zhang and Cai (2010)), the damage state at an arbitrary point in the material is represented by a properly defined integrity variable. The integrity parameter reaches its critical value at fracture. According to this notion, a crack in a fracture process can be modeled with the concept of a completely damaged zone in the vicinity of the crack tip. Namely a crack can be represented by a region where the integrity state has attained to its critical state, i.e., by the completely damaged zone (CDZ). Then the development of the crack and its preceding damage can be elucidated by analyzing the local states of stress, strain and damage.

The CDZ may be interpreted as the zone of critical decrease in the effective area due to damage development. Inside the completely damaged zone the damage involved reaches its critical value (for instance, the damage parameter reaches unity) and a complete fracture failure occurs. In view of material damage stresses are relaxed to vanishing (Stepanova and Igonin (2014), Stepanova and Adulina (2014), Stepanova and Yakovleva (2014)). Therefore, one can assume that the stress components in the CDZ equal zero. Outside the zone damage alters the stress distribution substantially compared to the corresponding non-damaging material. Well outside the CDZ the continuity parameter is equal to 1. Therefore asymptotic remote boundary conditions have the form

4. Conclusions

Asymptotic crack-tip fields in damaged materials are developed for a stationary plane stress crack under mixed mode loading. The asymptotic solution is obtained by the use of the similarity variable. On the basis of the similarity variable and the self-similar representation of the solution the near crack-tip stress, creep strain rate and continuity distributions are given. It is shown that meso-mechanical effect of damage accumulation near the crack tip results in new intermediate stress field asymptotic behavior and requires the solution of nonlinear eigenvalue problems. To attain eigensolutions a numerical scheme is worked out and the results obtained provide the additional eigenvalues of the HRR problem. By the use of the method proposed the whole set of eigenvalues for the mode crack in a power law material under mixed mode loading can be determined. The self-similar solutions are based on the idea of the existence of the completely damaged zone near the crack tip. The higher order terms of the asymptotic expansions of stresses, creep strain rates and continuity parameter allowing to construct the contours of the completely damaged zone in the vicinity of the crack tip are derived and investigated.

5. References

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