Application of Analytical Design of Aggregated Regulators Method to Nutrient-Phytoplankton-Zooplankton Models

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***Abstract*—The article discusses application of theory of synergetic control to mathematical biology in the form of Nutrient-Phytoplankton-Zooplankton models. Three different objectives for the control function are considered: achieving a predetermined phytoplankton population size, achieving proportionality between phytoplankton population size and quantity of the nutrients, achieving proportionality between phytoplankton (prey) and zooplankton (predator) population sizes. For each of the objectives an analytical form of the control function is obtained.**

Keywords— NPZ, ADAR, control theory, mathematical biology, algae blooms, phytoplankton, zooplankton

I. Introduction

One of the approaches to better understand and predict algae blooms is the use of computer simulation. It can be used to predict algae blooms and determine exact conditions for the bloom to start, through the analysis of analytical solutions of the system of differential equations. It is based on mathematical modeling of the dynamics of a water body, a formal description of the relationships between various types of the living entities. One of the well-known models for describing interactions between phytoplankton (bloom source) and zooplankton (a natural predator for phytoplankton) is the Nutrient-Phytoplankton-Zooplankton (NPZ) model [1, 2]. Such models are derived from a Lotka-Volterra system of differential equations [3, 4]. In this article, the authors consider an NPZ-model as a basic model for algae blooms.

II. NPZ-model

A. System of equations

$$\begin{cases} \dfrac{dx_1}{dt} = ax_2 + bx_3 - cx_1x_2 \\ \dfrac{dx_2}{dt} = cx_1x_2 - dx_2x_3 - ax_2 \\ \dfrac{dx_3}{dt} = dx_2x_3 - bx_3 \end{cases} \quad (1)$$

The NPZ model (1) is the most general model that reflects the interaction of phytoplankton, zooplankton and nutrients [5]. It is based on the general Lotka-Volterra (predator-prey) model. Here x_1, x_2, x_3 are the densities of nutrition, phytoplankton and zooplankton, respectively.

III. Control

We introduce control to the model (1) in order to increase its robustness and make it more suitable for describing complex processes in the aquatic ecosystem. Control can be interpreted as an external action caused by abiotic factors: changes in the flow of water, its chemical composition, plankton migration and others. Thus, the change in population density will occur not only due to natural processes, but also due to other environmental factors. Analytical design of aggregated regulators (ADAR) method it used to add control to the NPZ model. The advantages of ADAR are the physical interpretability of the resulting control equation and its robustness [6, 7].

IV. NPZ-model with control

A. Objective: achieve a given phytoplankton population size.

Let us formulate the first control objective – achieve a given phytoplankton population size:

$$\psi(t) = x_2(t) - x_2^*,$$

where x_2^* is the target value of phytoplankton population size.

Application of ADAR method allows us to add control u to the model (1), so it takes the following form:

$$\begin{cases} \dfrac{dx_1}{dt} = ax_2 + bx_3 - cx_1x_2 + u \\ \dfrac{dx_2}{dt} = cx_1x_2 - dx_2x_3 - ax_2 \\ \dfrac{dx_3}{dt} = dx_2x_3 - bx_3 \\ u = -\dfrac{\psi^{(I)}}{T_1} - ax_2 - bx_3 + cx_1x_2 + \dfrac{d\varphi}{dt} \\ \psi(t) = x_2(t) - x_2^* \\ \psi^{(I)}(t) = x_1(t) - \varphi(t) \\ \varphi(t) = \dfrac{-\dfrac{\psi(t)}{T_2} + dx_2(t)x_3(t) + ax_2(t)}{cx_2(t)} \\ \dfrac{d\varphi(t)}{dt} = \dfrac{\left(-\dfrac{f_2}{T_2} + d\left(f_2x_3(t) + x_2(t)f_3\right) + af_2\right)x_2}{cx_2^2(t)} - \\ \qquad -\dfrac{\left(-\dfrac{\psi(t)}{T_2} + dx_2(t)x_3(t) + ax_2(t)\right)f_2}{cx_2^2(t)} \end{cases} \quad (2)$$

B. *Objective: achieve proportionality between phytoplankton population and nutrient flow without expanding the phase space*

We use a different control goal to obtain a new behavior of the system [8-11]. The new behavior assumes an equilibrium state between phytoplankton and food at a given ratio with a known maximum capacity of the environment. New control objective is:

$$\psi(t) = x_2(t) + \rho x_1(t) - q$$

Where ρ is the ratio and q is the maximum carrying capacity. NPZ-based model with the new control objective:

$$\begin{cases} \dfrac{dx_1}{dt} = ax_2 + bx_3 - cx_1x_2 + u \\ \dfrac{dx_2}{dt} = cx_1x_2 - dx_2x_3 - ax_2 \\ \dfrac{dx_3}{dt} = dx_2x_3 - bx_3 \\ u = -\dfrac{\psi}{\rho T_1} - \dfrac{f_2}{\rho} - ax_2 - bx_3 + cx_1x_2 \\ \psi(t) = x_2(t) + \rho x_1(t) - q \end{cases} \qquad (3)$$

C. *Objective: achieving proportional populations of prey and predator with the expansion of the phase space*

The control objective applied in the previous section was building a system regarding the proportionality of phytoplankton and their diet. System with a new control objective allows to consider the case of proportional coexistence of phytoplankton and zooplankton:

$$\psi(t) = x_2(t) + \rho x_3(t) - q$$

A system with control according to the given objective can be written as follows:

$$\begin{cases} \dfrac{dx_1}{dt} = ax_2 + bx_3 - cx_1x_2 + u \\ \dfrac{dx_2}{dt} = cx_1x_2 - dx_2x_3 - ax_2 \\ \dfrac{dx_3}{dt} = dx_2x_3 - bx_3 \\ u = -\dfrac{\psi^{(I)}}{T_1} - ax_2 - bx_3 + cx_1x_2 + \dfrac{d\varphi(t)}{dt} \\ \dfrac{d\varphi(t)}{dt} = \dfrac{x_2\left(-\dfrac{f_2 + \rho f_3}{T_2} + d\left(f_2x_3 + x_2f_3\right) + af_2\right)}{cx_2^2} - \\ -\dfrac{x_2\left(\rho\left(d\left(f_2x_3 + x_2f_3\right) - bf_3\right)\right)}{cx_2^2} - \\ -\dfrac{\left(-\dfrac{\psi}{T_2} + dx_2x_3 + ax_2 - \rho\left(dx_2x_3 - bx_3\right)\right)f_2}{cx_2^2} \\ \varphi(t) = \dfrac{-\dfrac{\psi}{T_2} + dx_2x_3 + ax_2 - \rho\left(dx_2x_3 - bx_3\right)}{cx_2} \\ \psi(t) = x_2(t) + \rho x_3(t) - q \\ \psi^{(I)}(t) = x_1(t) - \varphi(t) \end{cases} \qquad (4)$$

V. CONCLUSION

Three models with control were produced for different objectives depending on the required behavior of the system. Numerical methods could be applied to the derived systems of equations in order to find approximate solutions, which we discuss in the full version of this paper. Analysis of the phase-space and stable points in the full version of the paper shows that it is possible to find exact conditions under which the coexistence of phytoplankton and zooplankton in the same habitat is possible.

The first objective achieved by model (2) allows to control phytoplankton population by changing the amount of available nutrients. The model could be used to predict the amount of nutrients required for a given phytoplankton population size as well as to predict the time it would take for the population size to stabilize at the given level.

The second objective achieved by model (3) allows to explore the dynamics of phytoplankton dependency on nutrients. It could be used to find phytoplankton population size established in the ecosystem if a given supply of nutrients is available.

The third objective achieved by model (4) makes it possible to control both phytoplankton and zooplankton populations through changes in the amount of nutrients available to phytoplankton.

Application of control theory and ADAR method to mathematical biology provides new ways to get insights into the underlying processes of species interaction. Numerical solutions of the derived systems of equations with control allow, for example, to find exact conditions for an algae bloom to take place.

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